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Statistical Morphology

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Abstract

The aim of this paper is to show that basic morphological operations can be incorporated within a statistical physics formulation as a limit when the temperature of the system tends to zero. These operations can then be expressed in terms of finding minimum variance estimators of probability distributions. It enables us to relate these operations to alternative Bayesian or Markovian approaches to image analysis.

We first show how to derive elementary dilations (winner-take-all) and erosions (loser-take-all). These operations, referred to as statistical dilations and erosion, depend on a temperature parameter $\beta = 1/T$. They become purely morphological as β goes to infinity and purely linear averages as β goes to 0. Experimental results are given for a range of intermediate values of β. Concatenations of elementary operations can be naturally expressed by stringing together conditional probability distributions, each corresponding to the original operations, thus yielding statistical openings and closings. Techniques are given for computing the minimal variance estimators.

1 Introduction

Image processing and computer vision requires nonlinear filtering. Two of the most successful approaches are mathematical morphology (see for example Matheron [9] and Serra [11]) and bayesian methods (see e.g. Geman and Geman [7]). These approaches arise from very different philosophies and are formulated very differently. Morphology involves a basic set of elementary operations which can be combined to perform nonlinear filtering, valley or peak extraction, edge detection, segmentation and other operations. The bayesian approach involves finding the best interpretation of the image assuming a probabilistic model. Most morphological operations can be implemented in such a way that they require little computation (see Vincent [13]) while the bayesian approach may require simulated annealing (Geman and Geman [7]) or continuation methods (Blake [2], Geiger and Girosi [4]). The bayesian approach can be related mathematically (Geiger and Girosi [4], Geiger and Yuille [5]) to a variety of other nonlinear filtering techniques.

This paper gives a reformulation of morphology, *statistical morphology*, which makes it more similar to the bayesian approach. Morphological operations are interpreted as the minimal variance estimators of probabilistic models. We show that these estimators can be calculated straightforwardly. Concatenations of operations are obtained from the laws of probability theory. This reformulation involves a temperature parameter T and standard morphology is obtained in the limit as $T \to 0$ while as $T \to \infty$, we obtain linear filtering.

The next section defines elementary statistical morphological operations. Then, in the third section, we show how to concatenate elementary operations and section 4 is concerned with the calculation of estimators. Section 5 shows experimental results of these statistical morphological operations. Lastly, in the sixth section, we briefly compare the morphological and bayesian approaches.

2 Basic Morphological Operations

Complex morphological operations are built out of concatenations of elementary operations. The most primitive operations are to replace the value at a certain pixel by the biggest, or smallest, value in a specific neighbourhood of that pixel. This corresponds to winner-take-all and loser-take-all respectively, i.e., in morphological terms, *dilations* and *erosions.*

More precisely, let I be a grayscale image. Morphologists often regard such objects as mappings from the discrete plane $\mathbb{Z}^2$ into $\mathbb{Z}$. The neighborhood mentioned above is here referred to as *structuring element.* It is a particular subset B of $\mathbb{Z}^2$, that, for the sake of simplicity, we suppose symmetric (i.e., $B = -B$) and finite throughout the paper. The dilation $\delta_B(I)$ and erosion $\varepsilon_B(I)$ of I by B are the grayscale images given by :

$$\begin{array}{ll} \delta_B(I) & \left(\begin{array}{lcl} \mathbb{Z}^2 & \longrightarrow & \mathbb{Z} \\ p & \longmapsto & \max\{I(q) \mid q \in B + p\} \end{array} \right. \\ \varepsilon_B(I) & \left(\begin{array}{lcl} \mathbb{Z}^2 & \longrightarrow & \mathbb{Z} \\ p & \longmapsto & \min\{I(q) \mid q \in B + p\}, \end{array} \right. \end{array} \tag{1}$$

where $B + p$ denotes the translation of B by p. Intuitively, dilation corresponds to taking maxima over a given neighborhood whereas erosion corresponds to taking minima.

Let us now rephrase these operations in terms more suited to the statistical physics framework. We here suppose that the input pixels are I_i and the neighbourhood of each pixel i is N_i. N_i is usually a translation of a given structuring element. But more generally, the neighborhood N_i considered may have different shapes at different locations. The derived operations are then no longer invariant by translation [11] and cannot be described with formulas (1). For pixel i, we will represent this neighbourhood by a symmetric matrix N_{ij}, such that $N_{ij} = 1$ if $j \in N_i$ and $N_{ij} = 0$ otherwise. We might also allow N_{ij} to be a general matrix, corresponding to alternative operations, this will not affect the analysis.

To obtain winner-take-all (dilation) for a specific pixel site i_0 we first define binary decision units $V_{i_0 j}$ such that $V_{i_0 j} = 1$ and $V_{i_0 k} = 0$ for $j \neq k$ means that the i_0^{th} site selects the j^{th} input as the winner. The following analysis is adapted from Geiger and Yuille [5].

We now define a cost function

$$E_w[V_{i_0 j}] = -\sum_j V_{i_0 j} N_{i_0 j} I_j. \tag{2}$$

Minimizing this energy with the constraint that $\sum_j V_{i_0 j} = 1$ (i.e. only one winner is allowed) will give $V_{i_0 w} = 1$, where $I_w > I_j$ for $j \neq w$, and $V_{i_0 j} = 0$ otherwise.

Following statistical physics we can define the Gibbs distribution [10]

$$P_w[V_{i_0 j}] = \frac{1}{Z} e^{-\beta E_w[V_{i_0 j}]}, \tag{3}$$

where $\beta = 1/T$ with T being the temperature and Z a normalization constant. It can be seen directly that, for finite temperature T, the lowest energy state is always the most probable. As $T \to 0$ the probability of being in this state tends to one. As $T \to \infty$ all states become equally likely.

It is straightforward to calculate

$$P_w[V_{i_0 k} = 1, V_{i_0 j} = 0, \ for\ all\ j \neq k] = \frac{e^{\beta N_{i_0 k} I_k}}{\sum_j e^{\beta N_{i_0 j} I_j}}. \tag{4}$$

In the limit as $\beta \to \infty$ (the zero temperature limit) only the lowest energy state will have non-zero probability of occuring, so we obtain the winner-take-all. In the infinite temperature limit, $\beta = 0$, we get $P_w[V_{i_0 j} = 1] = 1/|N_{i_0}|$ for all j where $|N_{i_0}|$ is the size of the neighbourhood.

We define the output of the system $O_w(\beta)$ to be

$$O_w(\beta) = \sum_k N_{i_0 j} I_k P_w[V_{i_0 k} = 1, V_{i_0 j} = 0, \; for \; all \; j \neq k] = \sum_k N_{i_0 j} I_k \frac{e^{\beta N_{i_0 k} I_k}}{\sum_j e^{\beta N_{i_0 j} I_j}} \tag{5}$$

(see next section for an interpretation of this definition).

Thus we obtain the morphological dilation in the limit as $\beta \to \infty$ and a linear filter corresponding to a spatial average of the inputs as $\beta \to 0$. Other linear filters could be obtained in the $\beta \to 0$ limit by varying the matrix $N_{i_0 j}$.

A similar analysis can be performed on erosions. We transform winner-take-all into loser-takes-all by altering the sign of the energy. This gives

$$E_l[V_{i_0 j}] = \sum_j V_{i_0 j} N_{i_0 j} I_j. \tag{6}$$

Repeating the analysis for this energy gives the output of erosions, $O_l(\beta)$, to be

$$O_l(\beta) = \sum_k N_{i_0 j} I_k P_w[V_{i_0 k} = 1, V_{i_0 j} = 0, \; for \; all \; j \neq k] = \sum_k N_{i_0 j} I_k \frac{e^{-\beta N_{i_0 k} I_k}}{\sum_j e^{-\beta N_{i_0 j} I_j}}. \tag{7}$$

Thus we have obtained formulae for *statistical dilation*, $O_w(\beta)$ for $\beta \geq 0$ given by (5), and *statistical erosion*, $O_l(\beta)$ for $\beta \geq 0$ given by (7). The standard dilation and erosion operations can be obtained in the limit as $\beta \to \infty$.

We now give an alternative interpretation. Observe that we have the identity

$$O_l(\beta) = O_w(-\beta). \tag{8}$$

This means that we could use one basic operation $O_w(\beta)$ and obtain dilation and erosion from it in the limits as $\beta \to \pm\infty$. Intuitively we could think of dilation as being erosion at negative temperature or vice versa.

Besides, this is another way of expressing the duality between statistical dilations and erosions : just like classical dilations and erosions with flat structuring elements, these operations satisfy for any $\beta \geq 0$:

$$\forall f : \mathbb{Z}^2 \longrightarrow \mathbb{Z}, \quad O_l(\beta)(-f) = -O_w(\beta)(f). \tag{9}$$

In other words, a statistical dilation of f reduces to a statistical erosion of $-f$.

Similarly, like their classical morphological counterparts, statistical erosions and dilations are increasing operations, i.e., they preserve the ordering relationships between images :

$$\forall f : \mathbb{Z}^2 \longrightarrow \mathbb{Z}, \quad f \leq g \Longrightarrow O_l(\beta)(f) \leq O_l(\beta)(g) \text{ and } O_w(\beta)(f) \leq O_w(\beta)(g). \tag{10}$$

However, one can be easily convinced that in the general case, neither operation is extensive or anti-extensive. Furthermore, unlike linear operations (which commute with algebraic additions) and morphological erosions and dilations (wich commute respectively with inf and sup [11]), statistical erosions and dilations generally commute with neither.

3 Concatenations of Operations

We now show how to combine elementary operations. First we reformulate these operations as probabilities of mappings from inputs $\{I_j\}$ to outputs $\{H_i\}$. We use the identity

$$P[H|I] = \sum_V P[H|V,I]P[V|I]. \tag{11}$$

For winner-take-all (statistical dilation) we set

$$P_w[V|I] = \frac{1}{Z} e^{\beta \sum_{ij} V_{ij} N_{ij} I_j}, \tag{12}$$

where we impose the requirement that for each i there exists a unique j such that $V_{ij} = 1$.

There are several possible choices for $P_w[H|V,I]$. The most obvious, the one we used in the previous section, is to assume that H is specified uniquely if the V's are given. This corresponds to

$$P_w[H|V,I] = \prod_i \delta(H_i - \sum_j V_{ij} N_{ij} I_j), \tag{13}$$

where δ is the Dirac delta function.

An alternative possibility comes from recalling that a delta function can be written as $\delta(x) = lim_{\sigma \to 0} \frac{1}{(2\pi)^{1/2}\sigma} e^{-x^2/2\sigma^2}$. Thus one could define $P_w[H|V,I] = \frac{1}{Z} e^{-\beta\lambda \sum_i (H_i - \sum_j V_{ij} N_{ij} I_j)^2}$, where λ is a fixed constant, from which we obtain equation (11) in the limit as $\beta \to \infty$. This choice needs to be further investigated and we will not use it in this paper.

Combining $P_w[H|V,I]$, given by (11), and $P_w[V|I]$ gives

$$P_w[H|I] = \frac{1}{Z} \sum_V e^{-\beta\{-\sum_{ij} V_{ij} N_{ij} I_j\}} \prod_i \delta(H_i - \sum_j V_{ij} N_{ij} I_j), \tag{14}$$

where Z is the normalization factor and the sum over the V maintains the restriction that $\sum_j V_{ij} = 1$ for each i.

It is now straightforward to see how to combine two different operations. Suppose we wish to start with a statistical dilation (winner-take-all) and follow it with a statistical erosion with the same neighborhood (loser-take-all) (this operation will be referred in the sequel as *statistical closing*). For the winner-take-all we define outputs H, decision units V, and $P_w[H|V,I]$ and $P_w[V|I]$ as above. For the loser-take-all we have inputs H, outputs O, decision units U, and define $P_l[O|U,H]$ and $P_l[U|H]$ to be

$$P_l[O|U,H] = \prod_i \delta(O_i - \sum_j U_{ij} N_{ij} H_j),$$
$$P_l[U|H] = \frac{1}{Z} e^{-\beta \sum_{ij} U_{ij} N_{ij} H_j}. \tag{15}$$

The combined operator, representing a statistical closing (statistical dilation followed by a statistical erosion), has $P_{lw}[O|I]$ given by

$$P_{lw}[O|I] = \sum_H P_l[O|H] P_w[H|I] = \sum_{V,U,H} P_l[O|U,H] P_l[U|H] P_w[H|V,I] P_w[V|I]. \tag{16}$$

To determine the output O we must specify an estimator for this probability distribution. We define the output to be the mean

$$\overline{O}(\beta) = \sum_{O} O P_{lw}[O|I]. \tag{17}$$

This corresponds (Gelb [6]) to the *minimal variance* (MV) estimator of the distribution or, in other words, to the estimator that chooses O_{mv} to minimize

$$M[O_{mv}] = \int [dO](O - O_{mv})^2 P_{lw}[O|I]. \tag{18}$$

This estimator is believed to be more robust than the *maximum likelihood* (ML) estimator, which would correspond to finding the output O_m that maximized $P_{lw}[O|I]$. In any case the MV reduces to the ML as $\beta \to \infty$ and will correspond to the morphological operations in this limit.

These rules of combination, given by (14), can clearly be extended to arbitrary concatenations of elementary operations by following the laws of probability theory. In the next section we show how to compute the MV estimators.

4 Computation of estimators

One of the advantages of morphological operations, compared with bayesian methods, is the ease of computation. In this section we describe techniques that can directly compute the statistical estimators.

We propose using the minimal variance estimator (Gelb [6]) of the probability distribution. Techniques for obtaining this estimator can be directly adapted from statistical physics.

We begin by considering the winner-take-all operation which has probability distribution given by (10) and (11). Using a standard technique from statistical physics (Parisi [10]) we will modify this distribution by including dummy variables $\{A_j\}$ and a corresponding term $e^{\sum_j A_j H_j}$ into the probability distribution $P_w[H|I]$. Note that we can recover the original distribution by setting $A_j = 0$ for all j. This gives

$$P_w[H|I, A] = \frac{1}{Z} \sum_{V} e^{-\beta\{-\sum_{ij} V_{ij} N_{ij} I_j\}} e^{\sum_j A_j H_j} \prod_i \delta(H_i - \sum_j V_{ij} N_{ij} I_j). \tag{19}$$

The normalization constant, or partition function, is a function of the $\{A_j\}$ given by

$$Z[A] = \int \prod_k dH_k \sum_V e^{-\beta\{-\sum_{ij} V_{ij} N_{ij} I_j\}} e^{\sum_j A_j H_j} \prod_i \delta(H_i - \sum_j V_{ij} N_{ij} I_j). \tag{20}$$

Differentiating with respect to the $\{A_j\}$ it follows directly that

$$\overline{H}_k[A, \beta] = \frac{1}{Z} \frac{\partial Z}{\partial A_k}, \tag{21}$$

where $\overline{H}_k[A, \beta]$ is the mean of the variable H_k with respect to the Gibbs distribution for fixed A and β. By setting $A = 0$ we obtain the mean of the original distribution $P_w[H|I]$.

It is straightforward to calculate $Z[A]$. First we integrate with respect to the $\{H_i\}$ to obtain

$$Z[A] = \sum_V e^{\beta \sum_{i,j} N_{ij} V_{ij} I_j} e^{\sum_j A_j \sum_k V_{jk} N_{jk} I_k} = \sum_V e^{\sum_{ij} (A_i + \beta) N_{ij} V_{ij} I_j}. \quad (22)$$

Next we sum over the possible states V imposing the constraint that for each i there exists a unique j such that $V_{ij} = 0$. This gives

$$Z[A] = \prod_i \sum_j e^{(A_i + \beta) N_{ij} I_j}, \quad (23)$$

and so by differentiating we obtain

$$\frac{1}{Z} \frac{\partial Z}{\partial A_i}[A = 0] = \sum_j N_{ij} I_j \frac{e^{\beta N_{ij} I_j}}{\sum_k e^{\beta N_{ik} I_k}}, \quad (24)$$

which agrees with the formula (5) given in section 2.

This procedure can be repeated for concatenations of operations. Consider the winner-take-all (statistical erosion) followed by a loser-take-all (statistical dilation) given in (7). To simplify the mathematics we consider the output O_k at a single site k.

We introduce a dummy field A_k as before and define

$$Z(A_k) = \sum_V \sum_U \int [dH] dO_k \prod_i \delta(H_i - \sum_j V_{ij} N_{ij} I_j) \delta(O_k - \sum_l U_{kl} N_{kl} H_l) e^{\beta \sum_{ij} N_{ij} V_{ij} I_j} e^{-\beta \sum_j N_{kj} U_{kj} H_j} e^{O_k A_k}. \quad (25)$$

Summing over the U's and V's, with the constraints that there is a unique l such that $U_{kl} = 1$ and that given i there exists a unique j such that $V_{ij} = 1$, yields

$$Z(A_k) = \int [dH] dO_k \{\sum_l \delta(O_k - N_{kl} H_l) e^{A_k O_k} e^{-\beta N_{kl} H_l}\} \{\prod_i \sum_j \delta(H_i - N_{ij} I_j) e^{\beta N_{ij} I_j}\}. \quad (26)$$

Integrating with respect to O_k gives

$$Z(A_k) = \int [dH] \{\sum_l e^{(A_k - \beta) N_{kl} H_l}\} \{\prod_i \sum_j \delta(H_i - N_{ij} I_j) e^{\beta N_{ij} I_j}\}. \quad (27)$$

Integrating with respect to the $\{H_l\}$ gives

$$Z(A_k) = \sum_q \sum_p e^{(A_k - \beta) N_{kq} N_{qp} I_p} e^{\beta N_{qp} I_p} \prod_{i \neq q} \{\sum_j e^{\beta N_{ij} I_j}\}. \quad (28)$$

Thus using the relation $\overline{O}_k = \frac{1}{Z(A_k)} \frac{dZ(A_k)}{dA_k}(A_k = 0)$ we can calculate the MV estimator as

$$\overline{O}_k = \frac{\sum_q \sum_p (N_{kq} N_{qp} I_p) e^{-\beta N_{kq} N_{qp} I_p} e^{\beta N_{qp} I_p} \prod_{i \neq q} \{\sum_j e^{\beta N_{ij} I_j}\}}{\sum_q \sum_p e^{-\beta N_{kq} N_{qp} I_p} e^{\beta N_{qp} I_p} \prod_{i \neq q} \{\sum_j e^{\beta N_{ij} I_j}\}}. \quad (29)$$

This expression is somewhat complex but it is interesting that we can write the output of a dilation followed by an erosion (at nonzero temperature) in one-shot as a differentiable input-output relation. Similar expressions can be derived from concatenations of additional operations.

One way to approximate this computation is to compute the means of each of the stages in order. More precisely, when computing the mean (MV estimator) of

$$P_{lw}[O] = \sum_{H} P_l[O|H] P_w[H|I] \tag{30}$$

we first compute the mean of H, $\overline{H} = \sum_H H P_w[H|I]$, and then compute the mean of O assuming inputs $\overline{H}$, i.e. $\sum_O O P_l[O|\overline{H}]$. In statistical physics terminology this could be considered a *mean field approximation* since we approximate the value of H by its mean value $\overline{H}$. It should be emphasized that the mean field approximation can be directly computed in this case, in constrast with many optimization problems where the mean field is defined by a consistency equation which might have several solutions (e.g. see Geiger and Yuille [5]).

The use of the mean field approximation will drastically simplify the computations, though at the cost of only approximating the minimal variance estimator. As $\beta \to \infty$, however, the mean field approximation will become exact and we will obtain the standard morphological operations. From this perspective one of the computational advantages of morphology is that the mean field approximation can be used and is easily computable.

5 Experimental Results

As mentioned earlier, when the temperature $T = 1/\beta$ is different from 0 or $+\infty$, statistical dilations and erosions are transformations which are in between moving averages (where the window is given by the structuring element) and morphological dilations and erosions. As an example, Fig 1 shows some intermediate steps between the blurring of grayscale image 1.a using a moving average linear filter and the dilation of this image. The window used in this example is a 5×5 square. One can notice that some of the blurred zones resulting from the moving average are progressively transformed into "plateaus". The fuzziness of image 1.b disappears progressively.

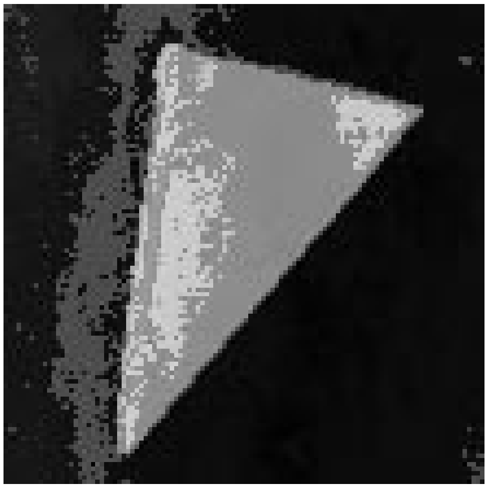
In the above example, a morphological dilation is opposed to a linear filter. But in fact, dilations are rarely used by themselves for filtering tasks. They usually constitute the first step involved in the computation of either a morphological closing or a closing by reconstruction or of more complex filters [12], or serve totally different purposes. It seems therefore more appropriate to observe the intermediate steps between an average and a morphological closing. This is done on Figure 2.

This figure illustrates well one of the potential uses of statistical morphological operations : in many cases, linear transformations provide efficient filtering, but tend to blur the images too much and destroy their edges. On the other hand, classical morphological filters preserve edges better, but perform poorly in some cases. Statistical openings and closings with $\beta \neq 0$ and $\beta \neq +\infty$ provide new filtering tools which, in some cases, retain the advantages of both the morphological and the linear approaches. For example on Fig. 2, the statistical closing with $\beta = 10$ seems to outperform the moving average as well as the morphological closing in terms of visual estimation of the filter.

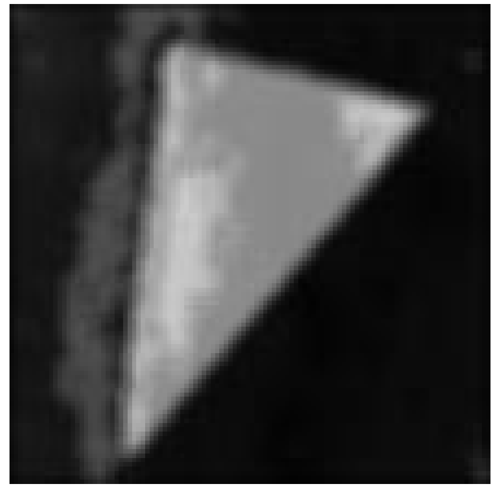
Another interest of statistical morphological operations concerns shape description and representation : when applied to a set or shape X represented via its associated characteristic function, statistical closings (resp. openings) give a whole range of intermediate steps between the "blurring" of the set and its morphological closing (resp. closing). This is illustrated by Fig. 3. Statistical morphology could therefore be at the basis of a general theory of shape and provide an alternative, or complement, to such work as Kimia's [8].

6 Comparisons with Bayesian Methods

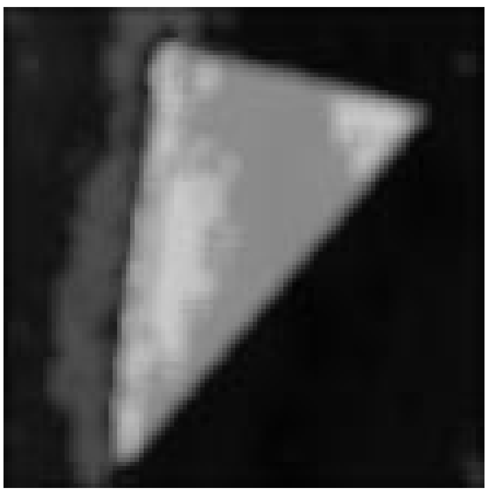
Dilation followed by erosion, or vice versa, gives a natural method for nonlinear smoothing of images. It is interesting to compare this approach with bayesian methods for nonlinear image smoothing. A classic example is the Geman and Geman (1984) model which, in one dimension, defines an energy function



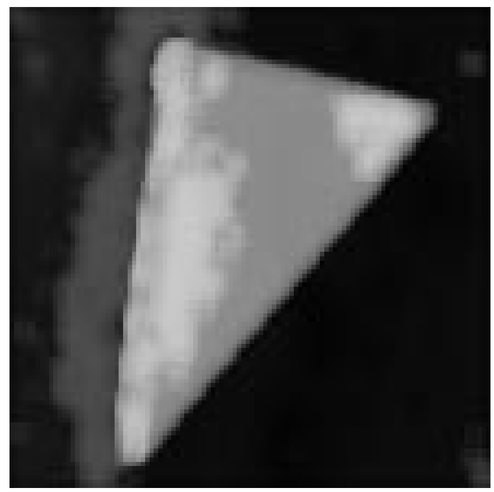
(a) original image



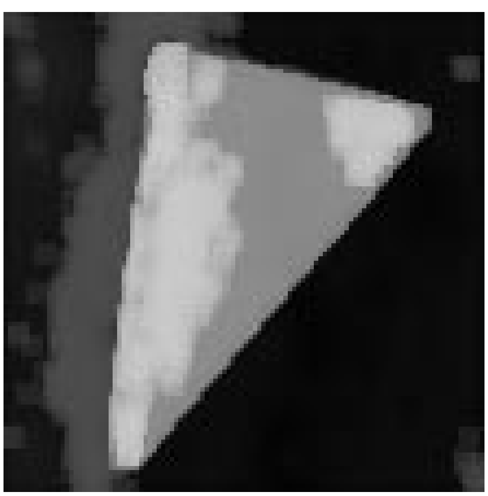
(b) statistical dilation, $\beta = 0$ *(moving average)*



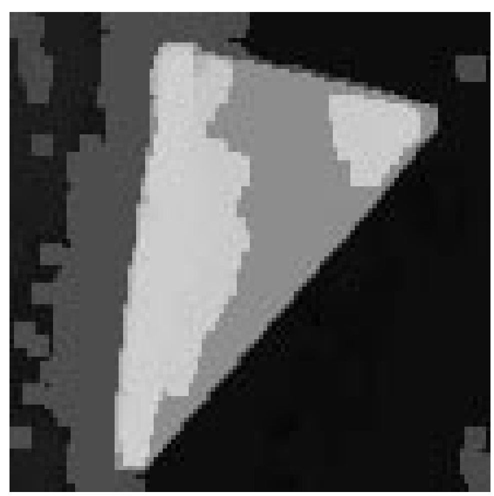
(c) statistical dilation, $\beta = 2$



(d) statistical dilation, $\beta = 5$

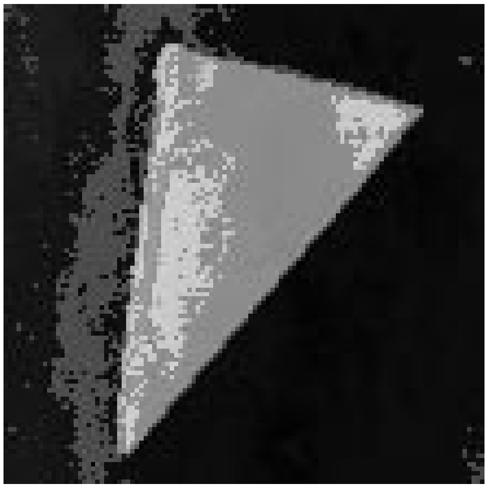


(e) statistical dilation, $\beta = 10$

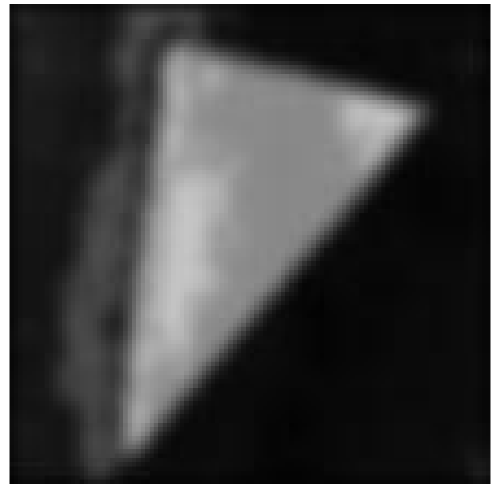


(f) statistical dilation, $\beta = +\infty$ *(standard dilation)*

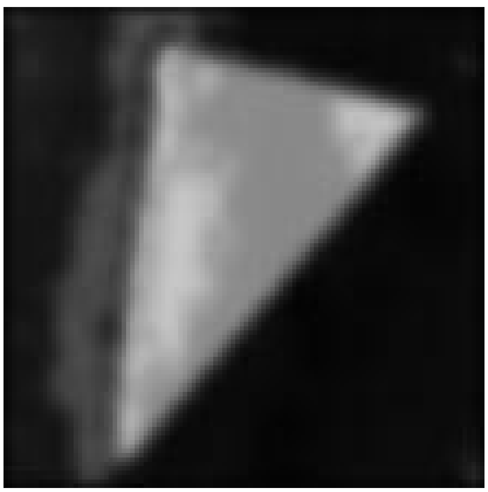
Figure 1 : Examples of statistical dilations of a grayscale image at different temperatures $T = 1/\beta$*. The structuring element used in this example is a* 5×5 *square.*



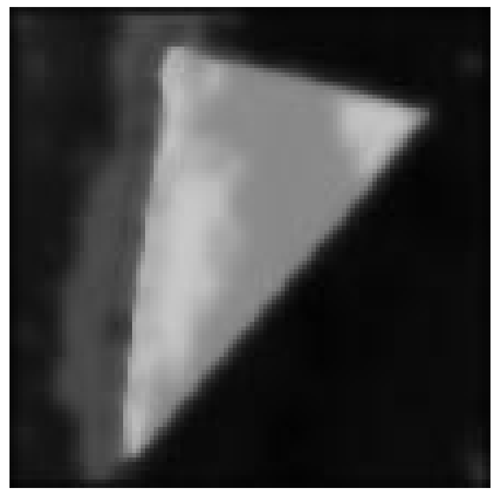
(a) original image



(b) statistical closing, $\beta = 0$ (moving average)

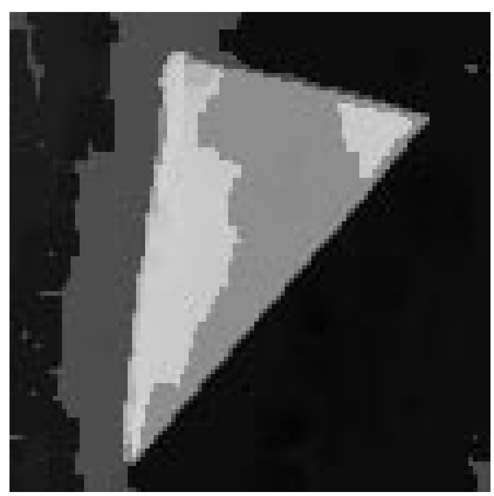


(c) statistical closing, $\beta = 2$



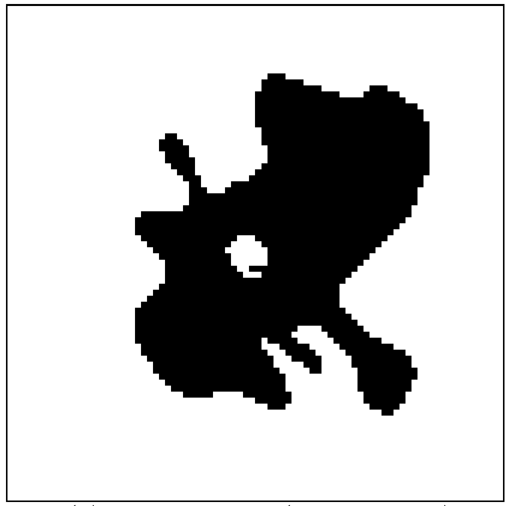
(d) statistical closing, $\beta = 5$

(e) statistical closing, $\beta = 10$

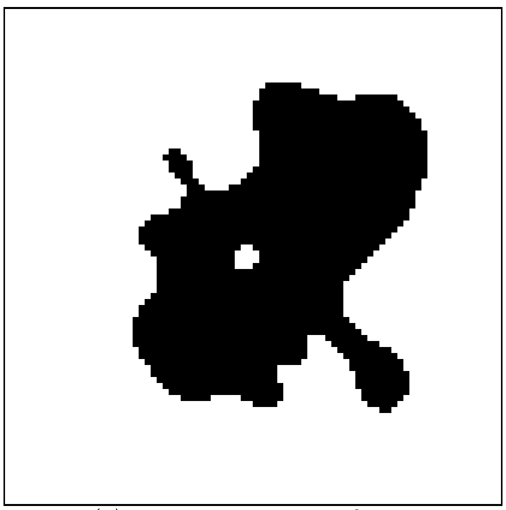


(f) statistical closing, $\beta = +\infty$ (standard closing)

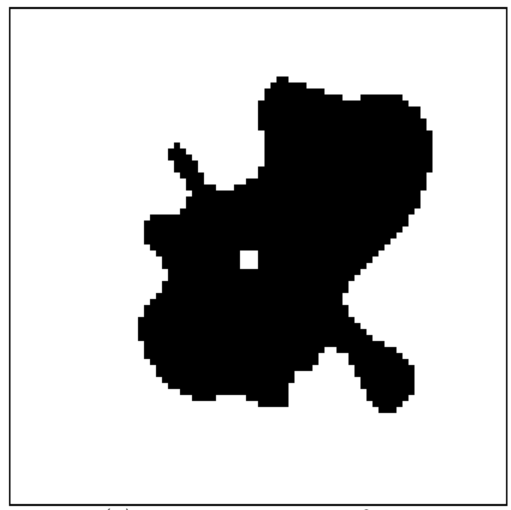
Figure 2 : *Examples of statistical closings of a grayscale image at different temperatures (same structuring element as for Fig. 1).*



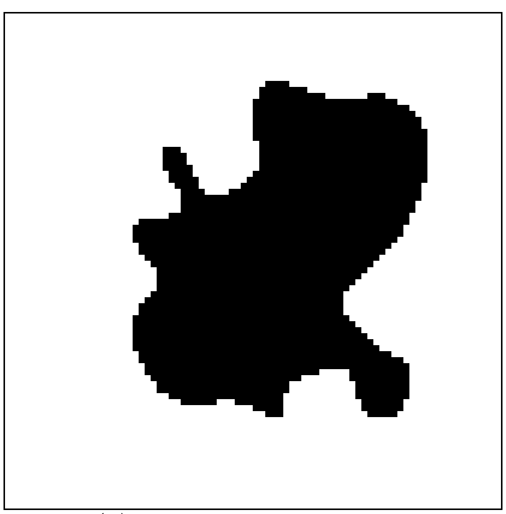
(a) original shape (binary image)



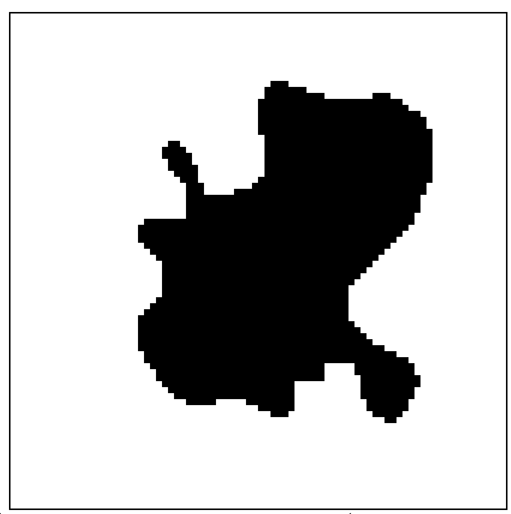
(b) statistical closing, $\beta = 0$



(c) statistical closing, $\beta = 1$



(d) statistical closing, $\beta = 2$



(e) statistical closing, $\beta = +\infty$ *(standard closing)*

Figure 3 : *Examples of statistical closings of a binary shape at different temperatures (used structuring element :* 5×5 *square). After each operation, the resulting image was thresholded at value 0.5.*

$$E[f,l] = \sum_i (f_i - d_i)^2 + \lambda \sum_i (f_{i+1} - f_i)^2 (1 - l_i) + \nu \sum_i l_i, \tag{31}$$

where $\{d_i\}$ is the data, $\{f_i\}$ represents the smoothed data and $\{l_i\}$ is a binary field. The theory proposes minimizing $E[f,l]$ with respect to f and l.

The second term in the cost function enforces smoothness between neighbouring sites i and $i+1$, but this smoothness can be broken by switching on the binary field l_i. This gives a form of nonlinear smoothing which preserves edges. It can be given a bayesian interpretation using the Gibbs distribution

$$P[f,l] = \frac{1}{Z} e^{-\beta E[f,l]}. \tag{32}$$

It was shown (Geiger and Girosi [4]) that using similar techniques to those in section (4), the binary fields $\{l_i\}$ can be eliminated to give

$$P[f] = \frac{1}{Z} e^{-\beta E_{eff}[f]}, \tag{33}$$

where

$$E_{eff}[f] = \sum_i (f_i - d_i)^2 - \frac{1}{\beta} \sum_i \log\{1 + e^{-\beta\{\lambda (f_{i+1} - f_i)^2 - \nu\}}\}. \tag{34}$$

The theory proposes finding the MV estimator for $P[f]$. This can be contrasted with the formula for erosion following dilation given by (25) where we identify f and d with O and I respectively.

There is no practical algorithm guaranteed to find the MV estimator of $P[f]$ (though in one dimension dynamic programming might succeed). Simulated annealing can be used (Geman and Geman [7]) and some heuristic techniques (Blake [2]) which can be related to mean field theory (Geiger and Girosi [4]) are highly effective in practice. Thus, computationally, these algorithms are far more complex than morpology.

The specific Geman and Geman model has difficulty dealing with shot-noise or eliminating small valleys or peaks in the intensity. This, however, can be fixed to some extent (Geiger and Yuille [5]) by adding an additional field that decides whether to accept or reject data.

7 Conclusion

We have described a framework for statistical morphology which includes standard morphology as a special case. Among other things, this has enabled us to formulate openings and closings as one-step operations, whose direct approximate computation is possible. It is hoped that such a framework will allow morphological methods to be directly compared with bayesian techniques.

We have also shown that the present approach provides a whole range of (statistical) operations in between moving averages and usual morphological transformations. In particular, statistical openings and closings should constitute interesting tools for filtering tasks where blurring has to be removed while preserving edges.

In the future, statistical physics methods could also become useful in the framework of morphological filtering [12] : they would indeed provide an alternative approach [3] for determining *optimal* morphological filters according to some constraints.

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