LECTURE NOTES ON MORPHOLOGICAL FILTERING

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Foreword

The following lecture notes constitute a course on the theory of morphological filtering. They have been initially written for the morphological part of the EURASIP short course *Median and Morphological Filtering in Image and Signal Processing,* given at the University of TAMPERE, Finland, in June 1989. They provide a simpler version of the theory introduced in *Image Analysis and Mathematical Morphology, Vol II (1988),* where the lattice of the grey-tone functions is particularly studied. However, all the key notions of morphological filtering are present and all theorems are proved.

After a first chapter of reminders, the next four ones are devoted to morphological filtering. Chapter 6 shows how to generate new classes of transformations from morphological filters. Finally, chapter 7, which is more oriented toward practice, presents a general approach to segmentation problems in mathematical morphology.

Chapter 1

Mathematical Morphology on complete lattices

1.1 Introduction

The main purpose of these lecture notes is to present the theory of morphological filtering, in a simpler version than the original one [19, Chapters 5 to 10]. It will occupy all the chapters of this text except the last two ones, which open other ways, and the current one, which is a reminder.

Before entering the technicalities, we would like to make clear, in a few words, the intuitions underlying our approach. The idea that governs convolution and more generally **linear** filtering is the preservation of addition. In acoustics, a linear amplifier respects the relative proportions of the various instruments. In optics also, convolution is a useful tool (one of the authors of this text is long-sighted, the other is short-sighted...). However, the visual signals are basically not compound by summation, unlike the acoustic vibrations on the drum of the ear. The world around us is not translucent. On the contrary, it is made of opaque objects that hide one another. The notion of *inclusion* is used to express this fundamental sort of relationships. Roughly speaking, it replaces, in the visual universe, the additivity of acoustic perception. Thus, the first prerequisite for any morphological filter ψ is that it should preserve inclusion, i.e. that it be an *increasing transformation:*

$$f \leq g \quad \Longrightarrow \quad \psi(f) \leq \psi(g), \tag{1.1}$$

where f and g are numerical functions on the Euclidean space or on a digital grid.

In summary, growth preserves contrast relations. Is it possible to conceive a filter that is both linear and increasing? Obviously not—the sup $f \vee g$ is different from the sum $f + g$ in that it has no inverse, and $(f \vee g) \wedge g$ cannot regenerate f as could $(f + g) - g$. Not only are both properties mutually exclusive, but there is a conflict in the very philosophy of the image analysis involved. Linearity often gives a group structure to convolutions, and thus allows us to deconvolve, i.e. to produce a clear image from a blurred one. This implies that there has been no information loss in the convolution that has produced the blur or the integration. On the contrary, an increasing transformation generally produces a *loss of information.* It is the reason why one cannot find an equivalent for Fourier space that would replace morphological filters by multiplications, or by any other reversible operation. More generally, we are not seeking to replace reversible operations by others, more or less adapted to a particular problem, but rather to accept this information loss as inevitable and to control it at best.

To achieve this, we shall add a second (and final) axiom to the definition of a morphological filter, namely **idempotence:**

$$\psi[\psi(f)] = \psi(f). \tag{1.2}$$

This condition stops the simplifying action of growth at the first stage (note that idempotent linear filters, such as band-pass or low-pass filters, turn out to be irreversible).

1.2 Algebraic framework of the complete lattices

Inclusion is a set oriented notion. The scenes under study may be modeled by sets, but also by grey-tone functions, by multi-spectral functions, by graphs, each of them acting either on the Euclidean space $\mathbb{R}^n$ or on digital ones, like $\mathbb{Z}^n$. All these situations share a common denominator formed by the two ideas which define the notion of a **complete lattice** $\mathcal{T}$ [4], namely:

1. there exists a partial ordering $\geq$ over $\mathcal{T}$,
2. for any (finite or infinite) family (A_i) in $\mathcal{T}$, there exists:
   - a smallest majorant $\vee A_i$ called the "sup" (for *supremum*),
   - a largest minorant $\wedge A_i$ called the "inf" (for *infimum*).

In particular, $\mathcal{T}$ posesses a greatest element, E, and a smallest one, $\emptyset$. In a lattice, any logical consequence of a choice of ordering remains true when we commute the symbols $\vee$ and $\wedge$, and $\leq$ and $\geq$. This is called the principle of **duality with respect to the order.**

Here is now a review of a few basic lattices:

1.2.1 Boolean lattices

Start from an arbitrary set E. Obviously, the set $\mathcal{P}(E)$ of the subsets of E, which is ordered for the inclusion relationship, is a complete lattice for the operations $\cup$ (union) and $\cap$ (intersection). Moreover, with each $X \in \mathcal{P}(E)$, there exists a unique $X^C \in \mathcal{P}(E)$, called the *complement* of X, such that:

$$X \cap X^C = \emptyset \qquad \text{and} \qquad X \cup X^C = E. \tag{1.3}$$

Finally, $\mathcal{P}(E)$ also satisfies the important property of **general distributivity** under which, for all $Y \in \mathcal{P}(E)$ and any family (X_i) of elements of $\mathcal{P}(E)$, we have:

$$(\bigcup X_i) \cap Y = \bigcup (X_i \cap Y), \tag{1.4}$$

$$(\bigcap X_i) \cup Y = \bigcap (X_i \cup Y). \tag{1.5}$$

1.2.2 Topological lattices

When E is a topological space, its open sets generate a complete lattice for the inclusion, where the sup coincides with the union and where $\inf(X_i)$ is the interior of $\bigcap X_i$. This lattice is not complemented. It satisfies the general distributivity of the type (1.4), but finite distributivity only of the type (1.5). Indeed, in the general case of an infinite family (X_i), we have

$$\begin{aligned} \overbrace{(\bigcup X_i)}^{\circ} \cap Y &= \overbrace{\bigcup (X_i \cap Y)}^{\circ}, \text{ but only} \\ \overbrace{(\bigcap X_i)}^{\circ} \cup Y &= \overbrace{\bigcap (X_i \cup Y)}^{\circ}. \end{aligned}$$

Similar structures are derived for the *closed* sets and the *compact* sets.

1.2.3 The convex lattice

The class of the convex sets of the Euclidean space $\mathbb{R}^n$ generates a complete lattice where the inf coincides with intersection and where the sup is the *convex hull.*

1.2.4 The partition lattice

In the set of the partitions of an arbitrary set E, we can introduce the following ordering: a partition A is smaller than a partition B when each class of A is included in a class of B. This leads to a lattice which is complete, but neither complemented nor distributive.

1.2.5 Function lattices

Let E be an arbitrary space. The class $\mathcal{F}$ of the real valued functions $f : E \longrightarrow \overline{\mathbb{R}}$ is obviously ordered by the relation: $f \leq g$ if for each $x \in E$, $f(x) \leq g(x)$ and constitutes a complete lattice. The sup and the inf are given by the relationships:

$$\begin{aligned} f &= \vee f_i \iff f(x) = \sup f_i(x), \quad \forall x \in E, \\ f' &= \wedge f_i \iff f'(x) = \inf f_i(x), \quad \forall x \in E. \end{aligned} \tag{1.6}$$

The lattice is completely distributive but not complemented. Rel. (1.6) implies that $f(x)$ may equal $+\infty$. However, if we want to restrict ourselves to bounded functions, it suffices to remark that the previous lattice is isomorphic (by anamorphosis) to

- either the class $\mathcal{F}'$ of the non negative functions $f : E \longrightarrow [0, +\infty]$,
- or the class $\mathcal{F}''$ of the functions $f : E \longrightarrow [0, 1]$.

Comments on functions and umbrae:

Is it possible to identify the function lattice $\mathcal{F}$ with the **set** class of the associated subgraphs, or umbrae? Remember that to every function $f : E \longrightarrow \overline{\mathbb{R}}$ (and more generally to every set in $E \times \overline{\mathbb{R}}$, see Fig. 1.1), correspond the two sets $U^+(f)$ and $U^-(f)$ of $E \times \overline{\mathbb{R}}$ defined by the relations:

$$U^+(f) = \{(x, z) \in E \times \overline{\mathbb{R}}, f(x) \leq z\}, \tag{1.7}$$

$$U^-(f) = \{(x, z) \in E \times \overline{\mathbb{R}}, f(x) < z\}. \tag{1.8}$$

Clearly,

$$f \leq g \iff U^+(f) \subseteq U^+(g) \iff U^-(f) \subseteq U^-(g),$$

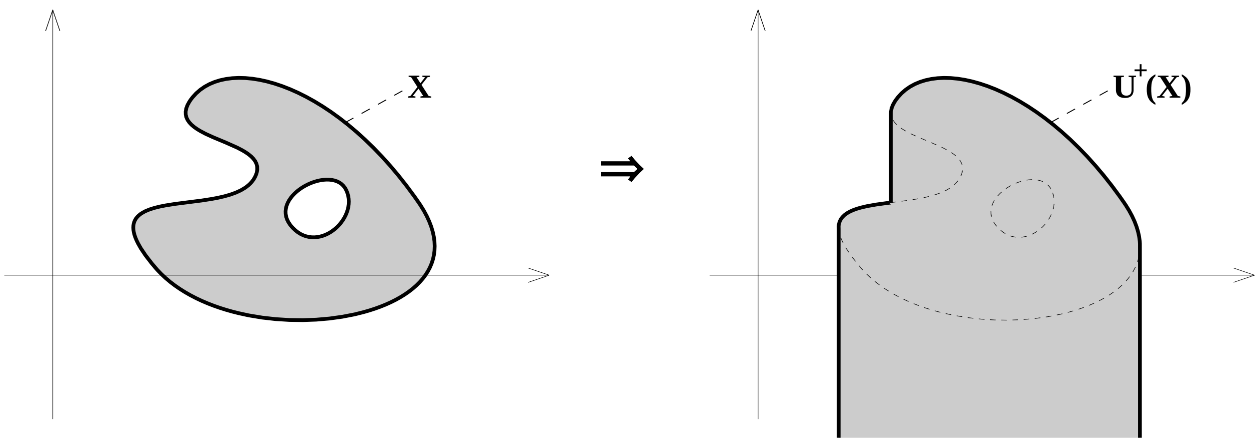


Figure 1.1: Umbra $U^+(X)$ of a set $X \subset \mathbb{R} \times \overline{\mathbb{R}}$.

and for finite families of operands, the three lattices are equivalent. Now, consider the *threshold mapping* defined as follows:

$$[\psi(f)](x) = \begin{cases} f(x) & \text{when } f(x) \geq 1, \\ -\infty & \text{otherwise.} \end{cases} \tag{1.9}$$

This operation is shown on Fig. 1.2.

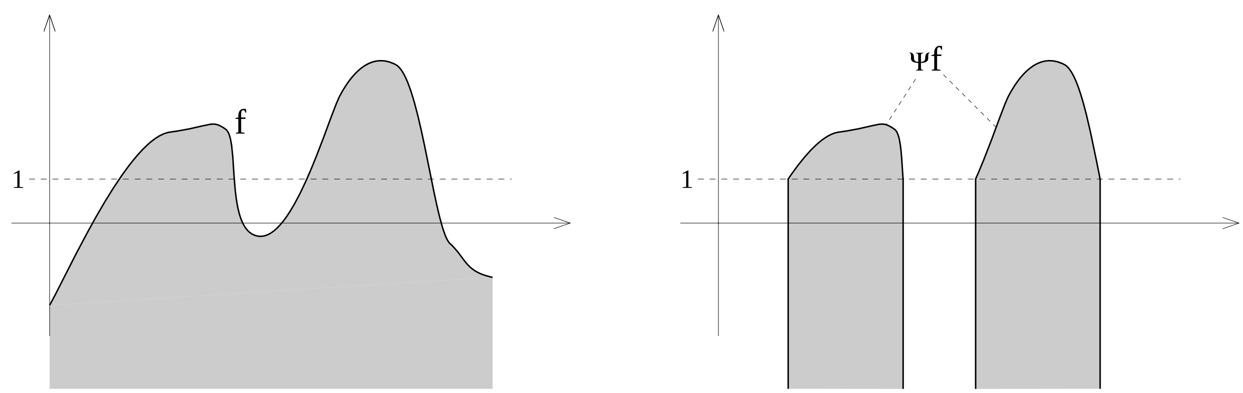


Figure 1.2: The threshold mapping ψ.

In set terms, the transformation ψ consists in intersecting the umbra $U^+(f)$ by the closed half space

$$E_1 = \{(x, y), x \in E, z \geq 1\},$$

and in taking the upper umbra of the result:

$$U^+(\psi(f)) = U^+[E_1 \cap U^+(f)] \cup E_{-\infty}. \tag{1.10}$$

If functions and upper umbrae are equivalent, then the two algorithms (1.9) and (1.10) must yield the same result. Let's apply both of them to the sup of the following family (see Fig. 1.3):

$$\begin{cases} f_i(x) = 1 - 1/i & \text{when } |x| \leq 1, \\ f_i(x) = -\infty & \text{otherwise.} \end{cases}$$

If the sup f of this family is understood in the sense of the function lattice, it is equal to:

$$\begin{cases} f(x) = 1 & \text{when } |x| \leq 1, \\ f(x) = -\infty & \text{otherwise,} \end{cases}$$

and according to the rel. (1.9), $\psi f = f$. But if the sup is understood in the upper umbrae lattice, i.e.

$$U^+(f) = \bigcup_i U^+(f_i),$$

then from rel. (1.10), we derive $U^+(\psi(f)) = E_{-\infty}$, i.e. $\forall x \in E, \psi f(x) = -\infty$. In other words, in the Euclidean case, the function lattice and the set oriented lattice of umbrae are not equivalent at all. Nevertheless, in the discrete case of functions $f : \mathbb{Z}^n \longrightarrow \mathbb{Z}$, the two approaches coincide and one can transpose the way of reasoning from sets to functions.

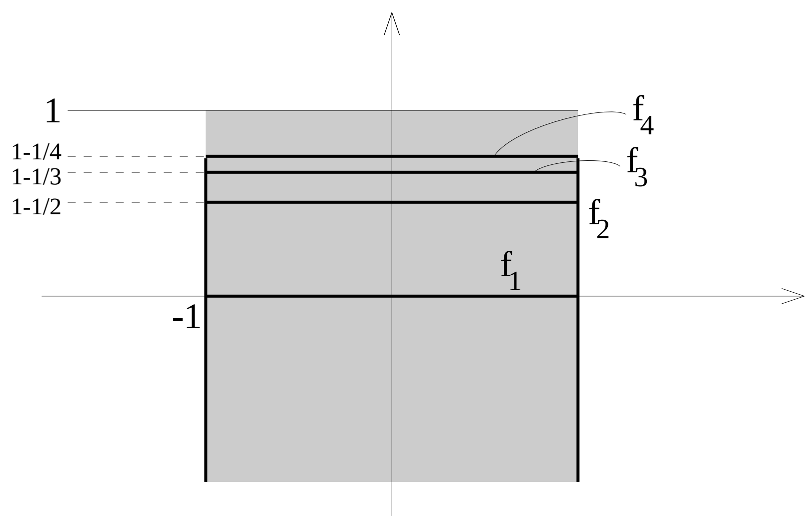


Figure 1.3: The family of functions (f_i).

1.2.6 Lattices of semi-continuous functions

A function f is *upper semi-continuous* (u.s.c.) when its umbra $U^+(f)$ is a closed set in $E \times \overline{\mathbb{R}}$. It is *lower semi-continuous* (l.s.c.) when the umbra $U^-(f)$ is an open set in $E \times \overline{\mathbb{R}}$. The use of semi-continuity becomes **strictly** necessary as soon as extrema are involved in the analysis under study, at least in continuous cases. For example, could we extract the maxima of the following function in $\mathbb{R}$ (see Fig. 1.4):

$$f(x) = \begin{cases} 1 - x^2 & \text{when } 0 < |x| < 1, \\ 0 & \text{when } |x| \geq 1 \text{ or } x = 0 \end{cases} \qquad ?$$

Actually, the maximum of such a function, although it is bounded, **does not exist.** Conversely, as soon as we refer to the "maximum" of a function over a continuous space, we implicitely introduce the requirement that it is u.s.c. (or l.s.c. when looking for minima).

In the lattice $\mathcal{F}_u$ of the u.s.c. functions, we have:

$$\inf_i f_i = \{f \in \mathcal{F}_u, U^+(f) = \bigcap_i U^+(f_i)\},$$

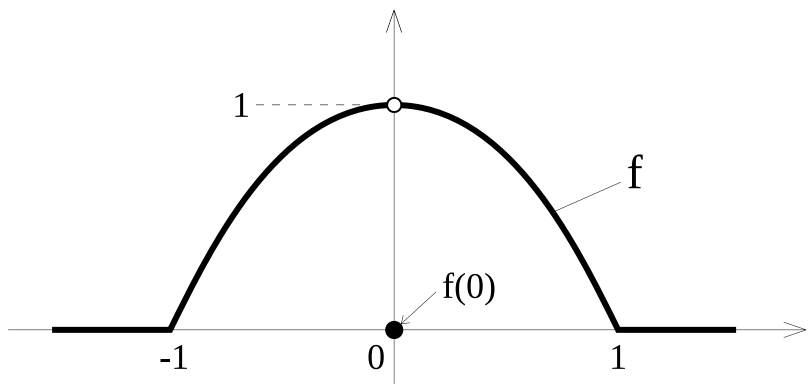


Figure 1.4: A function without a maximum (it is not u.s.c.).

$$\sup_i f_i \quad = \quad \{f \in \mathcal{F}_u, U^+(f) = \overline{\bigcup_i U^+(f_i)}\},$$

a notation which shows that the lattice $\mathcal{F}_u$ and that of the closed upper umbrae are *isomorphic* (in this case, the identification between sets and functions works.).

1.2.7 multi-spectral images

Clearly, multi-spectral images generate a lattice where the ordering, the sup and the inf are taken channel by channel, (e.g. $I_1 \leq I_2$ when the inequality holds, separately, for each channel.).

1.3 Erosion, dilation

In the same way that linear image processing puts the emphasis on the transformations that commute with addition, morphology naturally stresses the transformations that commute with the sup or, by duality, with the inf. This results in the following definition:

Definition 1.1 *Let $\mathcal{T}$ be a complete lattice. The mappings from $\mathcal{T}$ into itself which commute with the* sup *(resp. the* inf*) are called dilations δ (resp. erosions ε):*

$$\delta(\vee X_i) = \vee \delta(X_i), \qquad \varepsilon(\wedge X_i) = \wedge \varepsilon(X_i), \qquad X_i \in \mathcal{T}, \tag{1.11}$$

with in particular $\delta(\emptyset) = \emptyset$ and $\delta(E) = E$.

The following theorem (see [19, page 24]) characterizes these operations:

Theorem 1.2 *Let $\mathcal{T}$ be a complete lattice. The classes of the dilations and of the erosions on $\mathcal{T}$ are two complete isomorphic lattices, which correspond to one another through the duality relation:*

$$\delta(X) \leq Y \iff X \leq \varepsilon(Y), \qquad X, Y \in \mathcal{T}, \tag{1.12}$$

To each dilation δ corresponds a unique erosion ε:

$$\varepsilon(X) = \vee\{B \in \mathcal{T}, \delta(B) \leq X\} \tag{1.13}$$

and to each erosion ε corresponds a unique dilation δ:

$$\delta(X) = \wedge\{B \in \mathcal{T}, \varepsilon(B) \geq X\}. \tag{1.14}$$

Not only dilations and erosions are themselves increasing mappings, but they generate two comprehensive classes of increasing mappings. Indeed, we have the following theorem [19, page 20]:

Theorem 1.3 *Any mapping $\psi : \mathcal{T} \longrightarrow \mathcal{T}$ such that $\psi(E) = E$ is increasing if and only if it can be written as*

$$\psi = \vee\{\varepsilon_B, B \in \mathcal{T}\},$$

with the erosions ε_B given by

$$\varepsilon_B(X) = \begin{cases} \psi(B) & \text{if } X \geq B, \\ \emptyset & \text{otherwise.} \end{cases}$$

(dual result for the dilation.)

1.4 Lattice of sets or boolean lattice

In this section and in the following, we would like to compare the two lattices which model the binary and the grey-tone images.

The first one is the boolean lattice $\mathcal{P}(E)$, where E is $\mathbb{R}^n$ or $\mathbb{Z}^n$ for example. We can look at mappings from $\mathcal{P}(E)$ into itself as extensions of mappings from E into $\mathcal{P}(E)$. In the following, lower-case letters such as x, y, a, b denote elements of E, or points, and capital letters denote elements of $\mathcal{P}(E)$. A point $x \in E$, when considered as an element of $\mathcal{P}(E)$, is written as $\{x\}$. The letter δ denotes the mapping $E \longrightarrow \mathcal{P}(E)$, which generates a dilation, as well as the dilation from $\mathcal{P}(E)$ into itself. We define a **structuring function** on $\mathcal{P}(E)$ as any mapping $\delta : E \longrightarrow \mathcal{P}(E)$. Then, we have [19, page 41]:

Theorem 1.4 *Let E be an arbitrary set. The datum of any mapping $\delta : E \longrightarrow \mathcal{P}(E)$ is equivalent to that of a dilation from $\mathcal{P}(E)$ into itself, again symbolized by δ, and defined by the relation*

$$\delta(X) = \bigcup_{x \in X} \delta(x), \qquad X \in \mathcal{P}(E). \tag{1.15}$$

Conversely, any dilation of $\mathcal{P}(E)$ into itself determines a unique structuring function $\delta : E \longrightarrow \mathcal{P}(E)$.

1.4.1 The three dualities

In any boolean algebra $\mathcal{P}(E)$, the duality w.r. to the complementation associates with each mapping ψ the operation $\psi^* = \Theta\psi\Theta$, where Θ designates the complement operator, as expressed by

$$\psi^*(X) = [\psi(X^C)]^C.$$

In the case of the dilation δ, we find

$$\delta^*(X) = [\bigcup_{x \in X^C} \delta(x)]^C = \bigcap_{x \in X^C} [\delta(x)]^C. \tag{1.16}$$

δ^*, which obviously commutes with the inf, is an erosion. $\delta^*(X)$ consists of the points that are not *descendant* from any point in the complement of X (that are not included in any $\delta(x)$ when $x \in X^C$), i.e:

1. those whose ancestors are all included in X,
2. those that do not have ancestors (a fixed part S, which remains the same for any set X).

We form another duality notion by operating on the structuring function with the **transposition** $\delta \longmapsto \check{\delta}$, i.e:

$$\check{\delta}(x) = \{y \in E, x \in \delta(y)\}.$$

The transpose $\check{\delta}(x)$ of $\delta(x)$ is made of the set of points from which x descends, hence $\check{\check{\delta}} = \delta$. The structuring function $\check{\delta}$ generates the dilation $\check{\delta}$:

$$\check{\delta}(X) = \{y \in E, \delta(y) \cap X \neq \emptyset\}. \tag{1.17}$$

From the two relations (1.16) and (1.17), we derive the links between these two dualities, and the basic one, namely $\delta \leftrightarrow \varepsilon$ (see rel. (1.12)):

$$\varepsilon = (\check{\delta})^* \qquad \check{\varepsilon} = \delta^* \qquad \varepsilon^* = \check{\delta}. \tag{1.18}$$

1.4.2 Translation invariance

We now assume that E is equipped with a translation (e.g. $E = \mathbb{Z}^n$ or $E = \mathbb{R}^n$), and that the dilation δ is translation invariant, i.e. is a t-dilation. In other words, the structuring function $\delta(h)$ at point h is deduced from that of the origin (denoted $\delta(o) = B$) by translation: $\delta(h) = B_h = \{B + h, b \in B\}$. The set B is called structuring element. We see from relation (1.15) that

$$\begin{aligned} \delta(X) &= X \oplus B = B \oplus X \\ &= \bigcup_{x \in X} B_x \\ &= \{b + x, x \in X, b \in B\} \\ &= \bigcup_{b \in B} X_b. \end{aligned}$$

The t-dilation δ is classically known as **Minkowski addition** between sets X and B. By duality under complementation, it gives

$$\delta^*(X) = \check{\varepsilon}(X) = \bigcap_{b \in B} X_b = X \ominus B$$

and by lattice duality:

$$\varepsilon(X) = \bigcap_{b \in \check{B}} X_b, \qquad \text{with } \check{B} = \{-b, b \in B\}.$$

Both operations are Minkowski subtractions of X by B and $\check{B}$ respectively. According to a classical result, any increasing t-mapping is a union of t-erosions, and also an intersection of t-dilations [12, page 221]. More precisely,

Theorem 1.5 *Let ψ be a translation invariant increasing mapping. Then, for any $X \in \mathcal{P}(E)$,*

$$\psi(X) = \bigcup\{X \ominus \check{B}, B \in \mathcal{P}(E), o \in \psi(B)\} = \bigcap\{X \oplus \check{B}, B \in \mathcal{P}(E), o \in \psi^*(B)\}.$$

1.4.3 Examples

We will now present some examples of dilations and erosions in the lattice of sets $\mathcal{P}(\mathbb{R}^2)$. Fig. 1.5 shows the dilation of a set X by a bipoint B, i.e. the set $X \oplus \check{B}$. Similarly, Fig. 1.6 illustrates the effect of an erosion of X by a segment S. On Fig. 1.7, the same set is dilated and eroded by a disc D (*Euclidean* dilation and erosion). The dilation by a disc is then compared, on Fig. 1.8, to that by an hexagon H of similar size. One can remark that many parts on the boundary of $X \oplus \check{H}$ are parallel to the vertices of H.

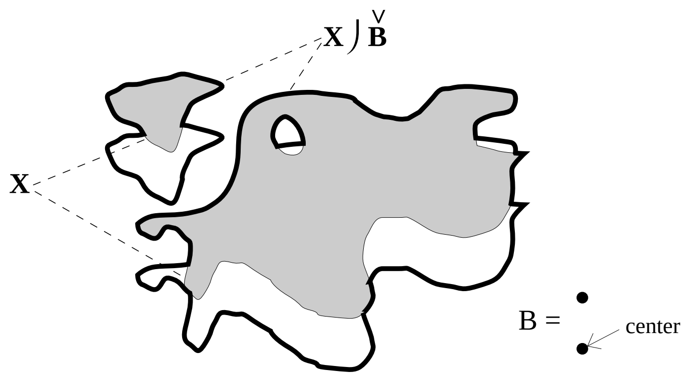


Figure 1.5: Dilation of a set X by a bipoint $\check{B}$.

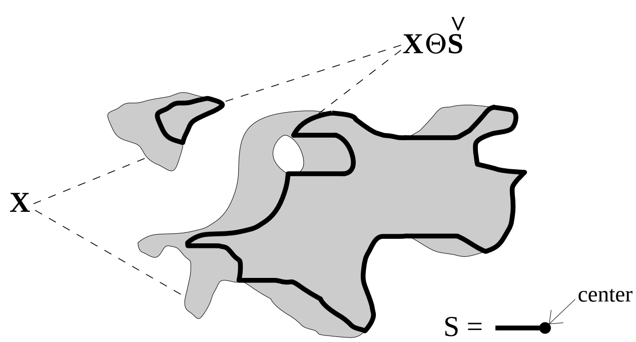


Figure 1.6: Erosion of a set X by a segment $\check{S}$.

Lastly, Fig. 1.9 illustrates the algorithm which is used for performing a *geodesic* dilation of a set Y inside a set X (see Chapter 7).

1.5 Lattices of functions

The lattice $\mathcal{F}(E, \overline{\mathbb{R}})$ of the functions $f : E \longrightarrow \overline{\mathbb{R}}$ shares several properties with the previous one, but it differs from $\mathcal{P}(E)$ by two major aspects:

1. it is not complemented,
2. when additions or subtractions are involved, they may lead to indetermination, of the type $+\infty - \infty$, since the range of variation is $\overline{\mathbb{R}}$.

We will now study $\mathcal{F}(E, \overline{\mathbb{R}})$ by following the same plan as for $\mathcal{P}(E)$.

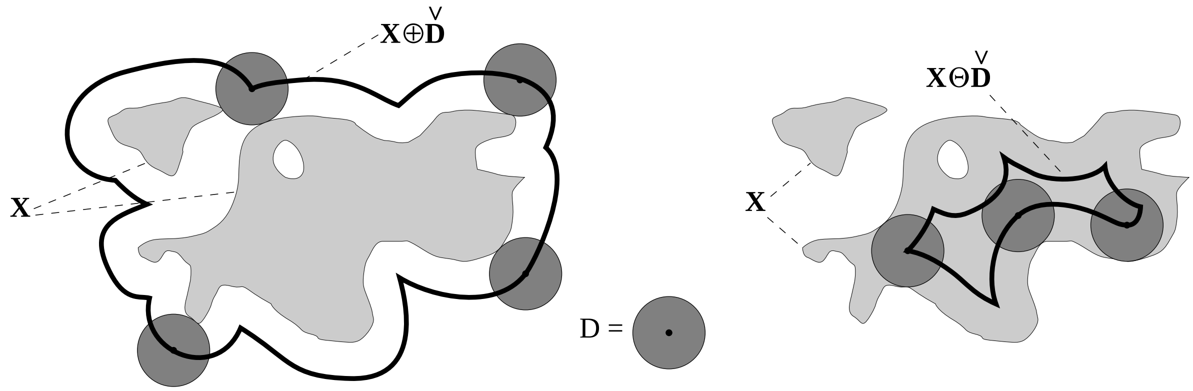


Figure 1.7: Dilation and erosion of X by a disc $D = \check{D}$.

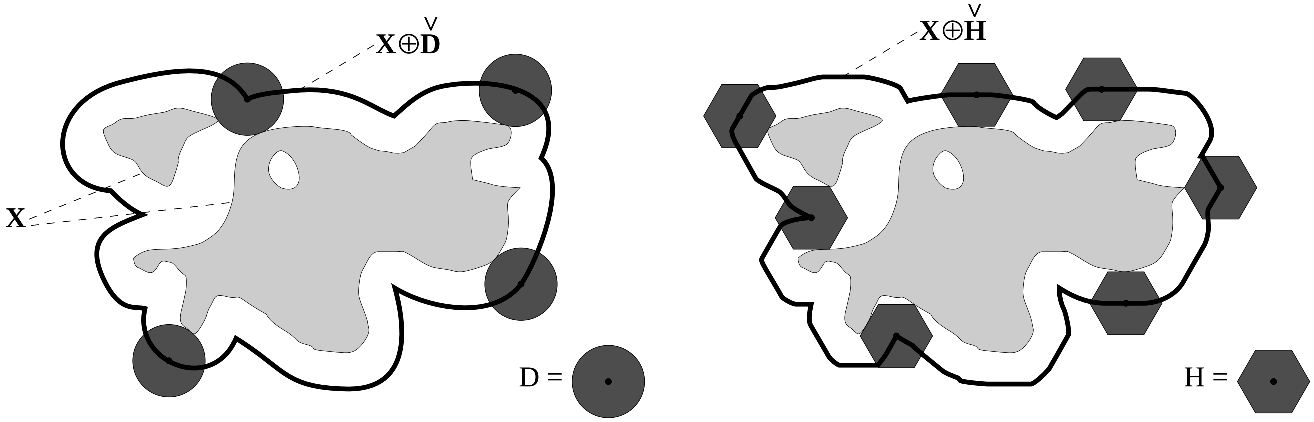


Figure 1.8: Comparison between the dilations of X by a disc and by an hexagon. Note that these structuring elements are symmetrical: $D = \check{D}$ and $H = \check{H}$.

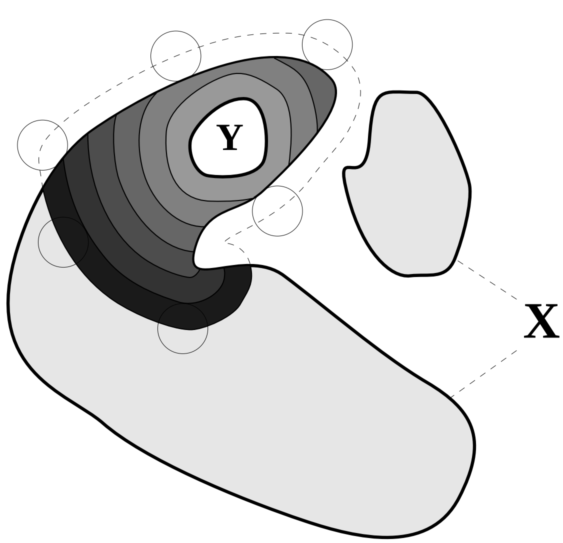


Figure 1.9: Successive geodesic dilations of set Y inside set X.

1.5.1 Generation of dilations from structuring functions

Call *impulse* $u_{h,z}$ a function whose value is z at point $h \in E$, and $-\infty$ elsewhere [6]:

$$\forall x \in E, \quad u_{h,z}(x) = \begin{cases} z & \text{when } x = h, \\ -\infty & \text{otherwise.} \end{cases}$$

The class $\mathcal{I}(E)$ of the impulses is equivalent to that of the points $(h, z) \in E \times \overline{\mathbb{R}}$. Clearly, any function $f \in \mathcal{F}(E, \overline{\mathbb{R}})$ is the sup of upper bounded impulses smaller than itself (just as a set is the union of the points it contains):

$$f = \sup\{u_{h,z}, h \in E, z < f(h)\}.$$

Introduce now a structuring function on $\mathcal{F}(E, \overline{\mathbb{R}})$ as any upper bounded mapping $\delta : \mathcal{I}(E) \longrightarrow \mathcal{F}(E, \overline{\mathbb{R}})$. We then have [19, page 185]:

Theorem 1.6 *any structuring function is equivalent to a dilation from $\mathcal{F}(E, \overline{\mathbb{R}})$ into itself, defined by the relation*

$$\delta(f) = \sup\{\delta(u_{h,z}), h \in E, z < f(h)\}. \tag{1.19}$$

Conversely, any dilation $\delta : \mathcal{F}(E, \overline{\mathbb{R}}) \longrightarrow \mathcal{F}(E, \overline{\mathbb{R}})$ induces a **unique** *structuring function obtained by restricting δ to $\mathcal{I}(E)$.*

1.5.2 Dualities

The transposition duality extends immediately to functions, by replacing points by impulses. The duality with respect to the complementation is replaced by all those given by the relation

$$\psi^*(f) = m - \psi(m - f), \tag{1.20}$$

as m spans the class of the real numbers. In practice, ψ often commutes with vertical shifts, i.e. $\psi(f + m) = \psi(f) + m$. Then, all the relations (1.20) are equal to $\psi^*(f) = -\psi(-f)$, and the three expressions (1.18) extend to functions.

1.5.3 Translation invariances

We can consider either a translation operation t'_h, by vector $h \in E$, or a translation operation $t_{h,z}$ by a vector $(h, z) \in E \times \overline{\mathbb{R}}$. The two corresponding formulas are:

$$\begin{aligned} (t_{h,z} f)(x) &= f(x - h) + z, \\ (t'_h f)(x) &= f(x - h). \end{aligned}$$

We shall focus on the t-invariant mappings, which are the most useful in practice. Saying that dilation δ is invariant with respect to translations is equivalent to saying that the *structuring function* δ is the same everywhere, i.e. if $g = \delta u_{0,0}$ is the transform of the origin-impulse, then $\forall x \in E, \delta u_{h,z}(x) = g(x - h) + z$. Then, the expression (1.19) of the dilation δ takes the following simpler form:

$$(\delta f)(x) = \sup\{g(x - h) + z, z < f(h), h \in E\}, \quad f \in \mathcal{F}(E, \overline{\mathbb{R}}).$$

Note that the operand $g(x-h)+z$ cannot take the undetermined form $+\infty - \infty$ since, for all h, x and z, each of the two numbers $g(x-h)$ and z is $< +\infty$. Hence, we have finally

$$(\delta f)(x) = \sup\{g(x-h) + f(h), h \in E\}, \qquad f \in \mathcal{F}(E, \overline{\mathbb{R}}). \tag{1.21}$$

The two dual erosions ε and $\check{\varepsilon}$ of δ are given by the following formulae:

$$\begin{aligned} (\varepsilon f)(x) &= \inf\{g(x+h) - f(h), h \in E\}, \\ (\check{\varepsilon} f)(x) &= \inf\{g(x-h) - f(h), h \in E\}. \end{aligned} \tag{1.22}$$

Similarly to theorem 1.5, any increasing mapping $\psi : E \longrightarrow \overline{\mathbb{R}}$ which is t-invariant may be decomposed into a sup of erosions as well as into an inf of dilations (same proof as for theorem 1.5).

1.5.4 Planar increasing mappings

An increasing mapping $\psi : \mathcal{F}(E, \overline{\mathbb{R}}) \longrightarrow \mathcal{F}(E, \overline{\mathbb{R}})$ is said to be planar, or flat, or again to be a *stack mapping* [25] when for any $z \in \overline{\mathbb{R}}$, the class $\mathcal{C}_z$ of the half cylinders of top level z is closed under ψ:

$$\begin{aligned} G_z \in \mathcal{C}_z &\iff \forall x \in E, G_z(x) = z \text{ or } G_z(x) = -\infty. \\ \psi \text{ is planar} &\iff \forall z \in \overline{\mathbb{R}}, \psi(\mathcal{C}_z) = \mathcal{C}_z \iff \forall z \in \overline{\mathbb{R}}, \forall G_z \in \mathcal{C}_z, \psi(G_z) \in \mathcal{C}_z. \end{aligned}$$

With any function $f \in \mathcal{F}(E, \overline{\mathbb{R}})$, we can associate its maximum cylinders $C_z(f)$:

$$C_z(f) = \sup\{G_z \in \mathcal{C}_z, G_z \leq f\}.$$

Then, the sup of the family $(C_z(f))_{z \in \overline{\mathbb{R}}}$ generates f:

$$f = \sup_{z \in \overline{\mathbb{R}}} C_z(f).$$

If ψ is increasing, we have, by growth

$$\psi C_z(f) \geq \sup\{\psi G_z, G_z \in \mathcal{C}_z, G_z \leq f\}.$$

Furthermore, if ψ is planar, then $\psi G_z \in \mathcal{C}_z$ and the inequality becomes

$$\psi C_z(f) \geq \sup\{G'_z, G'_z \in \mathcal{C}_z, G'_z \leq \psi f\}.$$

Now, by construction, $\psi C_z(f)$ is itself one of the G'_z, so that the above relation turns out to be an equality. In this equality, the right member is nothing but $C_z \psi(f)$. Finally:

$$\psi C_z(f) = C_z \psi(f),$$

hence:

$$\psi(f) = \sup_{z \in \overline{\mathbb{R}}} C_z \psi(f) = \sup_{z \in \overline{\mathbb{R}}} \psi C_z(f). \tag{1.23}$$

This relation is illustrated by Fig. 1.10.

In other words, the planarity of the mapping ψ allows us to process f threshold by threshold. A series of results derive directly from the key relation (1.23), namely:

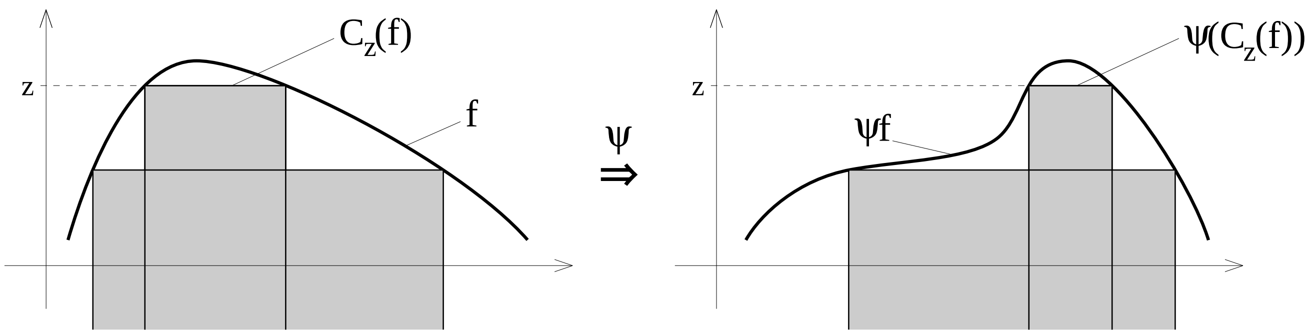


Figure 1.10: An example of a planar increasing mapping.

Theorem 1.7 *Let ψ be a planar increasing mapping from $\mathcal{F}(E, \overline{\mathbb{R}})$ into itself. Then:*

1. *the class of step functions with k levels (k arbitrary) is closed under ψ,*
2. *ψ commutes with anamorphosis.*

An anamorphosis is a strictly increasing and continuous point mapping $s : \overline{\mathbb{R}} \longrightarrow \overline{\mathbb{R}}$. For example, if $s = \exp$, then for any planar increasing mapping ψ, we have:

$$\psi \exp(f) = \exp \psi(f), \qquad f \in \mathcal{F}(E, \overline{\mathbb{R}}),$$

i.e. "vertical" and "horizontal" dimensions are treated **independently.**

A basic example of a planar increasing mapping consists in the dilation of f by a function b which is equal to 0 on its support B and to $-\infty$ everywhere else. Then, the relation (1.21) becomes:

$$(\delta f)(x) = \sup\{f(x-h), h \in B\} = (f \oplus B)(x). \tag{1.24}$$

By duality w.r. to lattice, relation (1.22) yields:

$$(\varepsilon f)(x) = \inf\{f(x+h), h \in B\} = (f \ominus \check{B})(x). \tag{1.25}$$

Fig. 1.11 shows an example of a planar dilation and of a planar erosion of a function f.

1.6 Digital implementations

In this section, we concentrate upon implementations of t-dilations (and t-erosions), which are the basic stones for building up more sophisticated algorithms.

When the dilation is planar, it is produced for functions in the same way as for sets. One has merely to replace union by sup and intersection by inf (e.g. refer to relation (1.24)). When the dilation is not planar, one can scan the successive levels of the structuring function, or use Steiner decomposition. In both cases, we shall use the following notation:

$$[f \oplus (1\ \mathbf{0})](x) = \sup\{f(x+1)+1, f(x)\}.$$

The number associated with each point denotes the altitude of the corresponding structuring function (here a function whose support is reduced to an horizontal doublet). When needed, a bold character is used for indicating the location of the origin. The elementary "spherical"—and centered—structuring functions are:

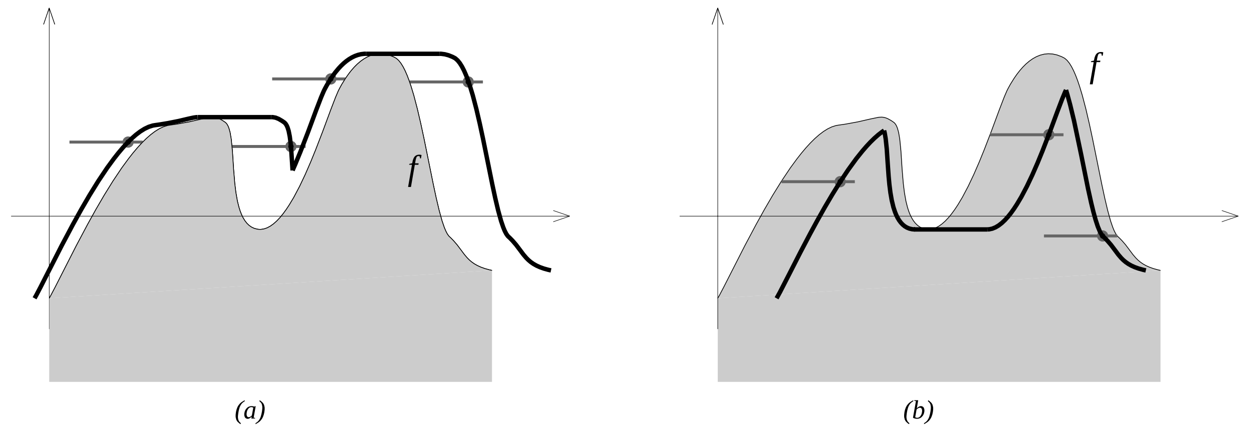


Figure 1.11: Dilation (a) and erosion (b) of a function f by a planar structuring element B.

- the cube: 9 pixels, on two successive levels
- the octahedron: 5 pixels, on two successive levels
- the rhombododecahedron: 9 pixels, on two successive levels
- the cuboctahedron: 9 pixels, on two successive levels.

They are represented on Fig. 1.12.

The elementary *rhombododecahedron* R can be represented (as in Fig. 1.13) by taking the spacing of the horizontal square grid to be $\sqrt{2}$.

The Steiner rhomb kR of size k is obtained by taking k dilations of R:

$$kR = \underbrace{R \oplus R \oplus \ldots \oplus R}_{k \text{ times}} = R^{\oplus k}.$$

As k increases, the difference between the Steiner rhomb and the ball becomes more apparent, but it is a simple matter to combine R with other Steiner polyhedra, such as the cuboctahedron, or simply with another Steiner rhomb, R^*, that is constructed at 45° to the first one (exactly as we construct octagons in $\mathbb{Z}^2$). This possibility is illustrated in Fig. 1.14.

The elementary *cuboctahedron* C does not lead to a sequence of segments. It has the decomposition shown in Fig. 1.15 (with the horizontal spacing being again equal to $\sqrt{2}$).

To dilate a function f by C, it suffices to perform

$$f_1 = f \oplus \begin{pmatrix} 0 & . & 0 \\ . & . & . \\ 0 & . & 0 \end{pmatrix}, \qquad f_2 = f \oplus \begin{pmatrix} . & 0 & . \\ 0 & 0 & 0 \\ . & 0 & . \end{pmatrix}$$

and then to compute the sup between f_1 and $f_2 + 1$:

$$f \oplus C = \sup\{f_1, f_2 + 1\}.$$

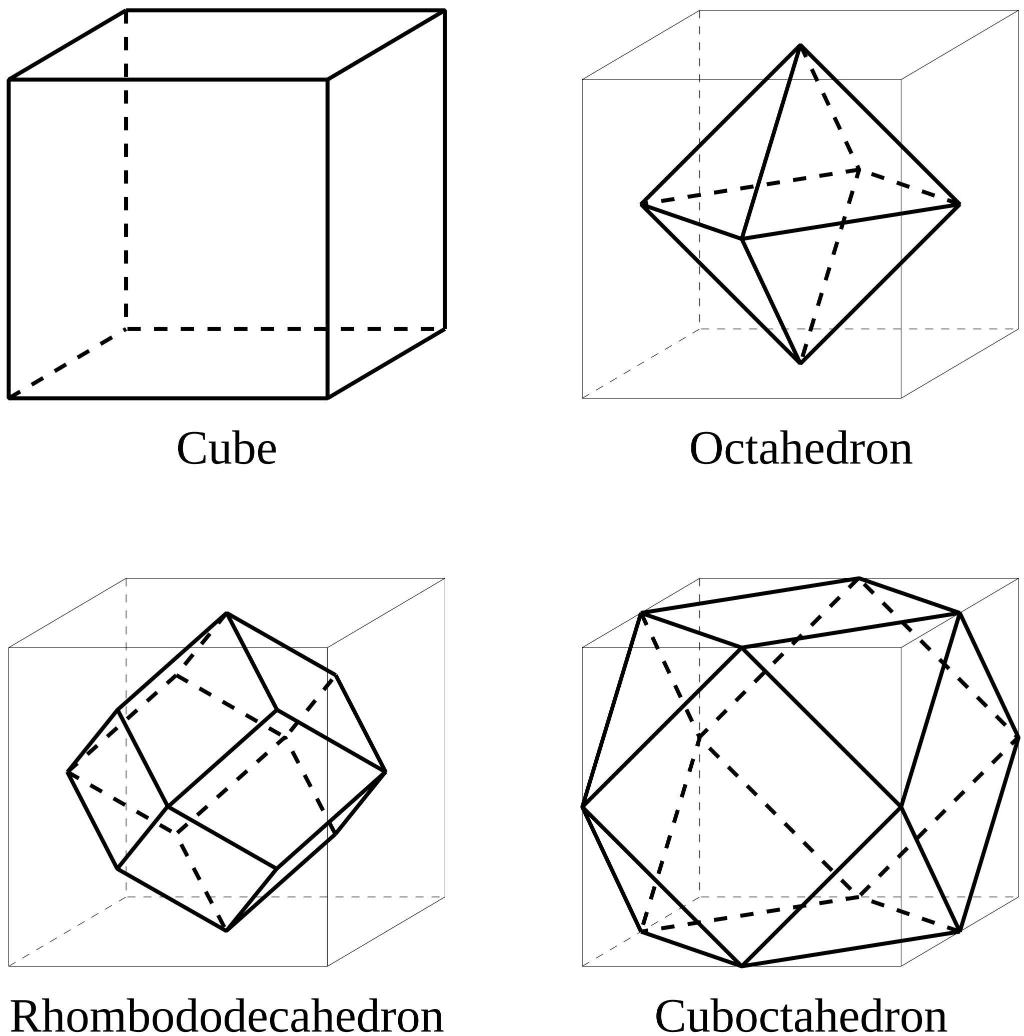


Figure 1.12: Basic spherical shapes in $\mathbb{Z}^3$. The plane $z = 0$ corresponds to the median horizontal section of the cube. The structuring functions derive from these sets by taking their umbrae.

$$\begin{matrix} . & . & . & . & . \\ . & 0 & 1 & 0 & . \\ . & 1 & 2 & 1 & . \\ . & 0 & 1 & 0 & . \\ . & . & . & . & . \end{matrix} \quad = \quad (\mathbf{1}\ 0) \oplus (0\ \mathbf{1}) \oplus \begin{pmatrix} \textbf{-1} \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ \textbf{-1} \end{pmatrix}$$

Figure 1.13: The elementary rhombododecahedron R (i.e. Steiner rhomb of $\mathbb{Z}^3$) and its decomposition in four dilations by segments. The complex shape of the polyhedron has been decomposed into four simple structuring functions, whose implementation is very simple and extremely efficient.

$$\underbrace{\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{array}}_{R} \oplus \underbrace{\begin{array}{ccccc} & & 0 & & \\ & 1 & . & 1 & \\ 0 & . & 2 & . & 0 \\ & 1 & . & 1 & \\ & & 0 & & \end{array}}_{R^*} = \underbrace{\begin{array}{ccccccc} & & 0 & 1 & 0 & & \\ & 1 & 2 & 2 & 2 & 1 & \\ 0 & 2 & 3 & 3 & 3 & 2 & 0 \\ 1 & 2 & 3 & 4 & 3 & 2 & 1 \\ 0 & 2 & 3 & 3 & 3 & 2 & 0 \\ & 1 & 2 & 2 & 2 & 1 & \\ & & 0 & 1 & 0 & & \end{array}}_{R \oplus R^*}$$

Figure 1.14: Dilation of the Steiner rhomb R by R^* (R^* is obtained in the same way as R, by four dilations by doublets).

$$\underbrace{\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{array}}_{C} = \underbrace{\begin{array}{ccc} 0 & . & 0 \\ . & . & . \\ 0 & . & 0 \end{array}}_{C_0} \cup \underbrace{\begin{array}{ccc} . & 1 & . \\ 1 & 1 & 1 \\ . & 1 & . \end{array}}_{C_1}$$

Figure 1.15: The elementary cuboctahedron C and its decomposition into two successive horizontal planes.

Chapter 2

Openings and closings

This chapter is a go-between from dilations to morphological filtering. Here, the two basic references are [19, chapters 7 & 17] and [12, chapters 1 & 5]. We shall see how, by looking for an *inverse* to the dilation—i.e. for an impossibility—we find a new operation, the *morphological closing,* whose three basic properties are extremely useful. We shall then try and keep these properties as axioms for the general concept of an (algebraic) closing. The notion of an *opening* is introduced by duality. It satisfies two of the three basic properties of the closing, that will become the two axioms of the morphological filtering in the next chapter.

2.1 Morphological opening and closing

Generally, in a complete lattice $\mathcal{T}$, the dilation $X \longrightarrow \delta(X)$ and the erosion $X \longrightarrow \varepsilon(X)$ do not admit inverses and there is no way for determining **one** element X from the images $\delta(X)$ or $\varepsilon(X)$. However, starting from a dilation and then performing the dual erosion (or the contrary), we always have either an upper, or a lower bound according to the situation at hand.

Indeed, if we take $\delta(X)$ for the set Y in relation (1.12), the left inclusion is satisfied, so $X \leq \varepsilon\delta(X)$, and by duality:

$$\delta \circ \varepsilon(X) \leq X \leq \varepsilon \circ \delta(X),$$

or in terms of operators:

$$\delta\varepsilon \leq I \leq \varepsilon\delta.$$

We say that $\varepsilon\delta$ is **extensive** (larger than the identity mapping) and that $\delta\varepsilon$ is **anti-extensive**. Both operations are also **increasing** as the product of increasing mappings. Now, $\varepsilon\delta \geq I$ implies, by growth, that $\delta\varepsilon\delta\varepsilon \geq \delta\varepsilon$, whereas $\delta\varepsilon \leq I$ implies the inverse inequality. Hence $\delta\varepsilon = \delta\varepsilon\delta\varepsilon$, i.e. is **idempotent** (as well as $\varepsilon\delta$, by duality). The three properties of $\varepsilon\delta$ characterize what is called a *closing,* in algebra, and those of $\delta\varepsilon$ an *opening.* We shall call these two operators *morphological* to indicate that they are generated from a dilation and its dual erosion, and we denote:

$$\gamma_m = \delta\varepsilon \qquad\qquad \varphi_m = \varepsilon\delta \tag{2.1}$$

Fig. 2.1 shows an example of a morphological opening and of a morphological closing of a set X in the plane. In this 2-D case, a morphological opening may remove three types of features: capes, isthma and islands. By duality, a morphological closing may fill gulfs, channels and lakes.

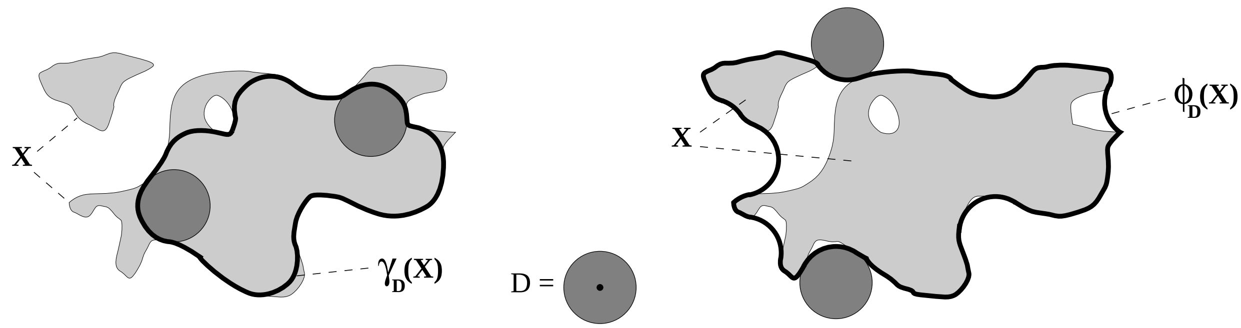


Figure 2.1: Examples of a morphological opening and of a morphological closing of a set X by a disc D.

Let $Z = \delta(X)$ be the dilation image of an arbitrary element $X \in \mathcal{T}$. We have:

$$\gamma_m(Z) = \delta\varepsilon\delta(X) \quad \begin{array}{ll} \geq \delta(X) & \text{by extensivity of } \varepsilon\delta \\ \leq \delta(X) & \text{by anti-extensivity of } \delta\varepsilon \end{array}$$

Hence $\gamma_m(Z) = Z$, i.e. Z belongs to the class $\mathcal{B}$ of the **invariant elements** of $\mathcal{T}$ under γ_m. Conversely if $Z \in \mathcal{B}$, then $Z = \delta(\varepsilon(Z))$ i.e. is the dilate of an element of $\mathcal{T}$. To summarize, we have the following theorem:

Theorem 2.1 *Given a dilation δ on lattice $\mathcal{T}$ and its dual erosion ε, the composition products $\gamma_m = \delta\varepsilon$ and $\varphi_m = \varepsilon\delta$ are respectively an opening and a closing on $\mathcal{T}$, called morphological. The invariance domain of the former is the image of $\mathcal{T}$ under δ and that of the latter forms the image of $\mathcal{T}$ under ε.*

Corollary 2.1 *Given $X \in \mathcal{T}$, $\gamma_m(X)$ is the smallest inverse image of X under ε, and $\varphi_m(X)$ is the largest one under δ.*

This corollary is illustrated by Fig. 2.2.

proof: Suppose that $Y \in \mathcal{T}$ is such that $\varepsilon(Y) = \varepsilon(X)$. Then, a fortiori, $Z = \varepsilon(X) \leq \varepsilon(Y)$ and thus, applying rel. (1.12), $\delta(Z) \leq Y$, or:

$$\gamma_m(X) \leq Y.$$

By duality, we have also

$$\forall Y \in \mathcal{T}, \delta(Y) = \delta(X) \Longrightarrow Y \leq \psi_m(X),$$

which completes the proof. □

Corollary 2.2 *If $\mathcal{B}$ and $\mathcal{B}'$ stand for the invariance domains of γ_m and φ_m respectively, then*

$$\gamma_m(X) = \vee\{B, B \in \mathcal{B}, B \leq X\} \tag{2.2}$$

$$\varphi_m X) = \wedge\{B, B \in \mathcal{B}', B \geq X\} \tag{2.3}$$

proof: From relation (1.13), we have

$$\begin{aligned} \gamma_m(X) = \delta(\varepsilon(X)) &= \delta(\vee\{B \in \mathcal{T}, \delta(B) \leq X\}) \\ &= \vee\{\delta(B), B \in \mathcal{T}, \delta(B) \leq X\}, \end{aligned}$$

but according to the theorem, $\mathcal{B} = \{\delta(B), B \in \mathcal{T}\}$. Hence, we get relation (2.2). As concerns relation (2.3), it has a dual proof. □

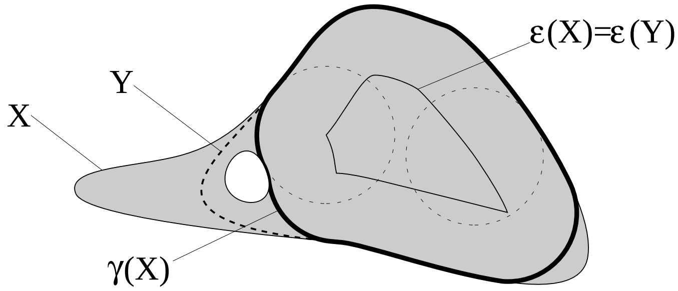


Figure 2.2: $\gamma_m(X)$ is the smallest element $Y \subseteq \mathcal{T}$ such that $\varepsilon(Y) = \varepsilon(X)$.

Example:

We have seen in § 1.5.4 that **planar** increasing mappings preserve vertical walls. Fig. 2.3 typically illustrates this point by showing the morphological opening of a 1-D function by an horizontal segment. Unlike this kind of opening, **circular** openings (i.e. openings with discs) do not preserve the vertical parts of the 1-D functions on which they act. In this case, Fig. 2.3 clearly indicates changes of slope. The same remarks apply in the 2-D case and the experimenter must choose between one approach or the other acording to his purpose. It should be noticed that "planar" structuring elements are most of the time preferred, since the computation the corresponding openings and closings can be done more efficiently than with 3-D structuring elements.

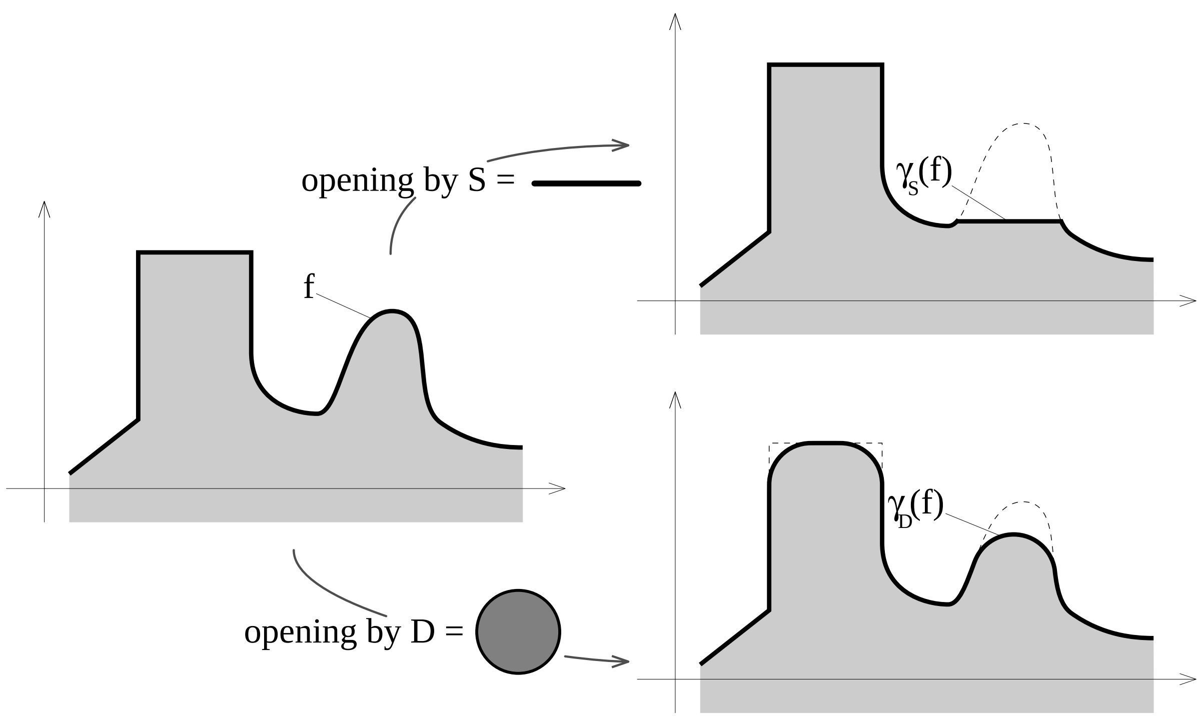


Figure 2.3: Comparison between the openings of a 1-D function by an horizontal segment S and by a disc D.

2.2 Algebraic openings and closings

The important corollary 2.2 directly associates an opening γ_m with its invariant elements, without referring to the intermediary erosion and dilation. Should it be also true for any algebraic opening γ, i.e. for any operation on $\mathcal{T}$ which is increasing, anti-extensive and idempotent? Let $\mathcal{B}$ be the invariance domain of such a γ, and B be an invariant element, $B \leq X$. Then (by growth) $B = \gamma(B) leq \gamma(X)$, hence $\gamma(X) \geq \vee\{B, B \in \mathcal{B}, B \leq X\}$. But $\gamma(X) \in \mathcal{B}$ (by idempotence) and $\gamma(X) \leq X$ (by anti-extensivity), therefore $\gamma(X)$ is one of the B of the right member. Thus, relation (2.2) is valid for any opening.

Conversely, start from an arbitrary part $\mathcal{B}_0$ of lattice $\mathcal{T}$ and let $\mathcal{B}$ be the class closed under union generated by $\mathcal{B}_0$. The operation defined by

$$\gamma(X) = \vee\{B, B \in \mathcal{B}_0, B \leq X\}$$

is increasing and anti-extensive. Moreover, $\gamma(X) = X$ iff $X \in \mathcal{B}$. The product $\gamma \circ \gamma$ is smaller than γ (growth and anti-extensivity), but also:

$$\begin{aligned} \gamma\gamma(X) &\geq \vee\{\gamma(B), B \in \mathcal{B}_0, B \leq X\} \\ &= \vee\{B, B \in \mathcal{B}_0, B \leq X\} \\ &= \gamma(X). \end{aligned}$$

We may therefore state the following:

Theorem 2.2 *An operation γ (resp. φ) on $\mathcal{T}$ is an opening (resp. a closing) if and only if there exists a class $\mathcal{B} \subseteq \mathcal{T}$, closed under union (resp. intersection) such that*

$$\begin{aligned} \gamma(X) &= \vee\{B, B \in \mathcal{B}, B \leq X\} \\ \varphi(X) &= \wedge\{B, B \in \mathcal{B}, B \geq X\}. \end{aligned}$$

$\mathcal{B}$ is the invariance domain *of γ (resp. φ).*

In other words, we can approach openings and closings either directly or via their invariance domains. Now, what about the *composition,* the *sup* or the *inf* of openings. Are they still operations of the same type? As far as sups are concerned, the anwer is yes. Indeed:

Theorem 2.3 *The sup of a family (γ_i) of openings is again an opening, whose domain of invariance is the class closed under union generated by the union of the $\mathcal{B}_i$ (invariance domains of the γ_i's).*

proof: Clearly, $\vee\gamma_i$ is increasing and anti-extensive. Furthermore, for all i, we have $\gamma_i \circ (\vee\gamma_i) \geq \gamma_i$. Therefore, $(\vee\gamma_i) \circ (\vee\gamma_i) \geq (\vee\gamma_i)$, and also the inverse inclusion, since $(\vee\gamma_i) \leq I$. This gives us the idempotence. The domain of invariance is determined as was done before. □

Unfortunately, the class of the openings is neither closed under $\wedge$, nor under composition. Consider for instance in $\mathbb{Z}$ the following set:

$$X = \texttt{..1111..111111..1111...}$$

and the two structuring elements

$$A = \texttt{.1.....1.} \quad \text{and} \quad B = \texttt{.11111.}$$

Denoting γ_A and γ_B, the associated morphological openings, we have:

$$\gamma_A(X) = X \quad \text{and} \quad \gamma_B(X) = \texttt{.111111.}$$

and

$$\gamma_B \circ \gamma_A(X) = \gamma_B(X) \neq \gamma_A \circ \gamma_B(X) = \emptyset.$$

Hence:

$$(\gamma_B \gamma_A)(\gamma_B \gamma_A) \neq (\gamma_B \gamma_A),$$

and

$$(\gamma_B \wedge \gamma_A)(\gamma_B \wedge \gamma_A)(X) = \emptyset \neq (\gamma_B \wedge \gamma_A)(X) = \gamma_B(X).$$

□

Let us quote a last result which clarifies the links between morphological and algebraic openings:

Theorem 2.4 *A mapping $\gamma : \mathcal{T} \longrightarrow \mathcal{T}$ is an opening if and only if it is the sup of a family (γ_i) of morphological openings. Moreover, if a translation is defined over $\mathcal{T}$, γ is translation invariant if and only if the γ_i are translation invariant (dual statement for the closings).*

proof: Easy, refer to [12, page 190], [18, page 161], [19, page 22]. □

2.3 (Non exhaustive) catalog of openings and closings

Although theorem 2.4 is heuristically deep, we may have difficulties in applying it directly, as the number of terms γ_i necessary for generating a given γ becomes prohibitive. Actually, there are four starting points for creating openings, namely:

- *the morphological openings,*
- *the trivial openings,*
- *the connected openings,*
- *the envelope openings.*

...plus any derivation obtained by cross-union of these various types. The first mode has already been developed. We will now present the other three.

2.3.1 Trivial openings

A criterion T is said to be *increasing* when, for all $X \in \mathcal{T}$:

$$\begin{cases} X \text{ satisfies } T \text{ and } Y \geq X & \Longrightarrow \quad Y \text{ satisfies } T, \\ X \text{ does not satisfy } T \text{ and } Y \leq X & \Longrightarrow \quad Y \text{ does not satisfy } T. \end{cases}$$

For example, in $\mathbb{R}^n$, for X to hit a given set A_0, as well as to have a Lebesgue measure larger than a given value λ_0 are both increasing criteria.

Proposition 2.5 *Given an increasing criterion T over lattice $\mathcal{T}$, the operation*

$$\gamma_1(X) = \begin{cases} X & \text{when } X \text{ satisfies } T, \\ \emptyset & \text{otherwise.} \end{cases}$$

(with $\gamma_1(\emptyset) = \emptyset$) is an opening called the trivial opening associated with criterion T.

2.3.2 Connected openingγ_x

We consider a boolean lattice $\mathcal{P}(E)$ and an arbitrary point $x \in E$. A part $\mathcal{C}$ of $\mathcal{P}(E)$ is called a **connected class** on $\mathcal{P}(E)$ when it satisfies:

(i) $\emptyset \in \mathcal{C}$ *and* $\forall x \in E$, $\{x\} \in \mathcal{C}$,

(ii) For every family (C_i) *in* $\mathcal{C}$, $\cap C_i \neq \emptyset \implies \cup C_i \in \mathcal{C}$.

One proves then [19, page 52] that the datum of a connected class $\mathcal{C}$ on $\mathcal{P}(E)$ is equivalent to the family of openings γ_x such that:

(iii) $\forall x \in E$, $\gamma_x(\{x\}) = \{x\}$,

(iv) $\forall A \subseteq E$, $x, y \in E$, $\gamma_x(A)$ *and* $\gamma_y(A)$ *are equal or disjoint, i.e.*

$$\gamma_x(A) \cap \gamma_y(A) \neq \emptyset \implies \gamma_x(A) = \gamma_y(A),$$

(v) $\forall A \subseteq E, \forall x \in E, \quad x \notin A \Longrightarrow \gamma_x(A) = \emptyset$.

At first sight, this theorem just indicates that the operation shown in Fig. 2.4 is an opening called *the connected component of A that contains point x*, which is somewhat obvious. But it also tells for instance that the operator:

$$\nu_x(A) = \begin{cases} (\gamma_x \circ \delta(A)) \cap A & \text{when } x \in A, \\ \emptyset & \text{otherwise.} \end{cases}$$

(where δ is a dilation by a disc) is again a connected opening associated with the connectivity shown in Fig. 2.5, which is less obvious [19, page 55].

2.3.3 Envelope openings

Consider a **finite** lattice $\mathcal{T}$ and an increasing mapping $\psi : \mathcal{T} \longrightarrow \mathcal{T}$. Then, for any $X \in \mathcal{T}$, the sequence $[X \cap \psi(X)]^n$ decreases with n, and finally stops for a certain n_0, since $\mathcal{T}$ is finite. The operator

$$\check{\psi} = (I \wedge \psi)^{n_0} \tag{2.4}$$

is therefore an opening. Moreover, if h is an opening smaller than ψ, then $h \leq I \wedge \psi$. Hence $h = h^n \leq (I \wedge \psi)^n$ for every n and thus $h \leq \check{\psi}$. In other words:

Theorem 2.6 *Let $\mathcal{T}$ be a finite lattice. Then, for every increasing mapping $\psi : \mathcal{T} \longrightarrow \mathcal{T}$, there exists an upper envelope $\check{\psi}$ of the openings which minorate ψ. It is itself an opening and is given by the relation*

$$\check{\psi} = (I \wedge \psi)^{n_0} \qquad \text{for a finite } n_0.$$

(dual result with $\hat{\psi} = (I \vee \psi)^{n_0}$.)

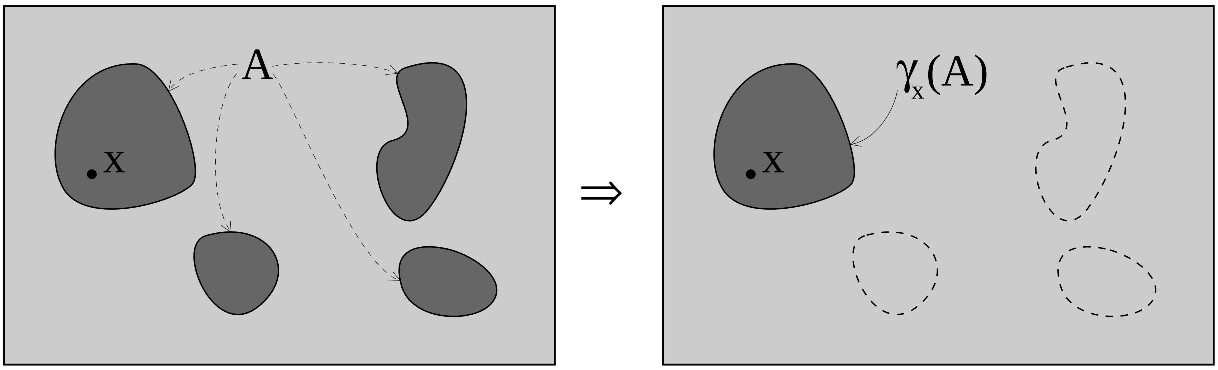


Figure 2.4: The opening called *the connected component of A that contains point x*.

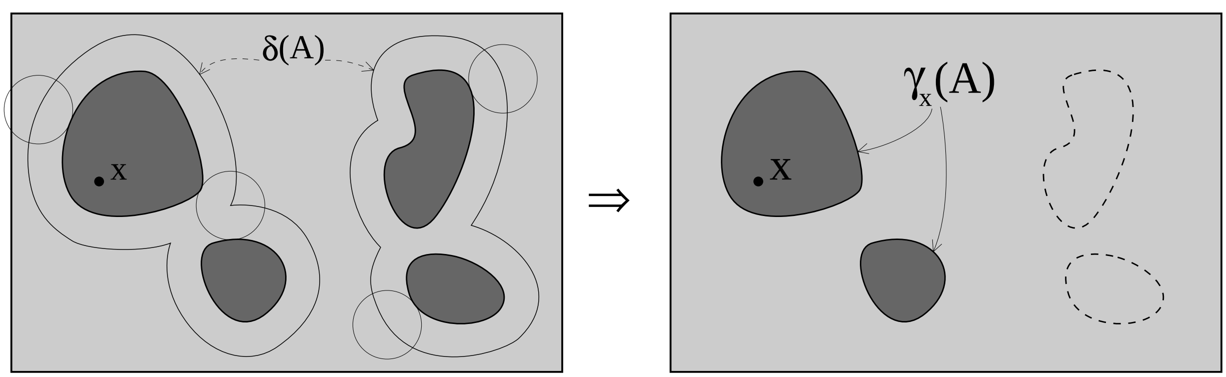


Figure 2.5: A less obvious connectivity notion, associated with the opening ν_x.

N.B: *(i)* The iterations may well stop at the first step. In § 4.3, the example of the rank-operators illustrates this point.

(ii) Under conditions which are always fulfilled in practice, theorem 2.6 extends to the lattice of the functions $f : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ [20].

2.4 Cross-unions of basic openings

2.4.1 Reconstruction algorithms (combination of types 2 and 3)

A series of algorithms are based on the same approach. For example, for 2-D binary images: *keep the connected components of X whose circular opening of size k is not empty and filter out the others.* The algorithm is extended to numerical functions via their horizontal sections, but can be directly implemented in terms of numerical operators. If ε_k and δ_k stand respectively for the isotropic erosion and dilation of size k (square, hexagonal, octogonal...) in $\mathbb{Z}^2$ (sets), as well as in $\mathcal{F}(\mathbb{Z}^2, \mathbb{Z}^+)$ (functions from $\mathbb{Z}^2$ into $\mathbb{Z}^+$), we can write:

$$Y = \varepsilon_k(X) \text{ (set case)} \qquad\qquad g = \varepsilon_k(f) \text{ (function case)},$$

and then:

$$\mu^{(1)}(Y) = \delta_1(Y) \cap X \qquad\qquad \mu^{(1)}(g) = \delta_1(g) \wedge f. \tag{2.5}$$

By iterations, we compute $\mu^{(2)}(Y) = \mu \circ \mu(Y)$, $\mu^{(3)}(Y) = \mu \circ \mu^{(2)}(Y)$,etc... The sequence of the $\mu^{(i)}$s increases till an idempotent limit, which provides the desired opening (see Fig. 2.6). The underlying methods, which are called *geodesic* ones, are presented in more details in § 7.3.

Remark that the same technique still yields an opening when replacing ε_k by any increasing criterion.

2.4.2 Annular opening (unions of types 1)

Consider the pair of points $B = \{o, b\}$, made of the origin o and a point b in direction α in $\mathbb{R}^2$ or in $\mathbb{Z}^2$. Clearly, the morphological opening γ_b with respect to B is equivalent to

$$\gamma_b = I \wedge \delta_{B'}$$

where $\delta_{B'}$ is the t-dilation by the bi-point $B' = \{-b; +b\}$. Now, make vary b in a certain domain D which does not contain the origin (e.g. three consecutive vertices of an hexagon centered on o, half a circle, ...) and take the sup γ:

$$\gamma = \vee\{\gamma_b, b \in B\} = I \wedge \{\vee\delta_{B'}, b \in D\}$$

i.e. since the dilation commutes with $\vee$:

$$\gamma = I \wedge \delta_{D\cup\check{D}} \tag{2.6}$$

where $\delta_{D\cup\check{D}}$ is the dilation by $D \cup \check{D} = \bigcup_{b\in D}\{-b; +b\}$. The effect of this *annular opening* γ is shown on Fig. 2.7. γ eliminates the components of a given set X as a function of their environment more than of their size or shape. On the example presented here, $D \cup \check{D}$ is taken to be a circle and γ eliminates the central particle without touching the others.

To illustrate the specific action of γ, we can compare it with the morphological opening γ' by a disc and with the union γ'' of the morphological openings by segments in various directions (see Fig. 2.7).

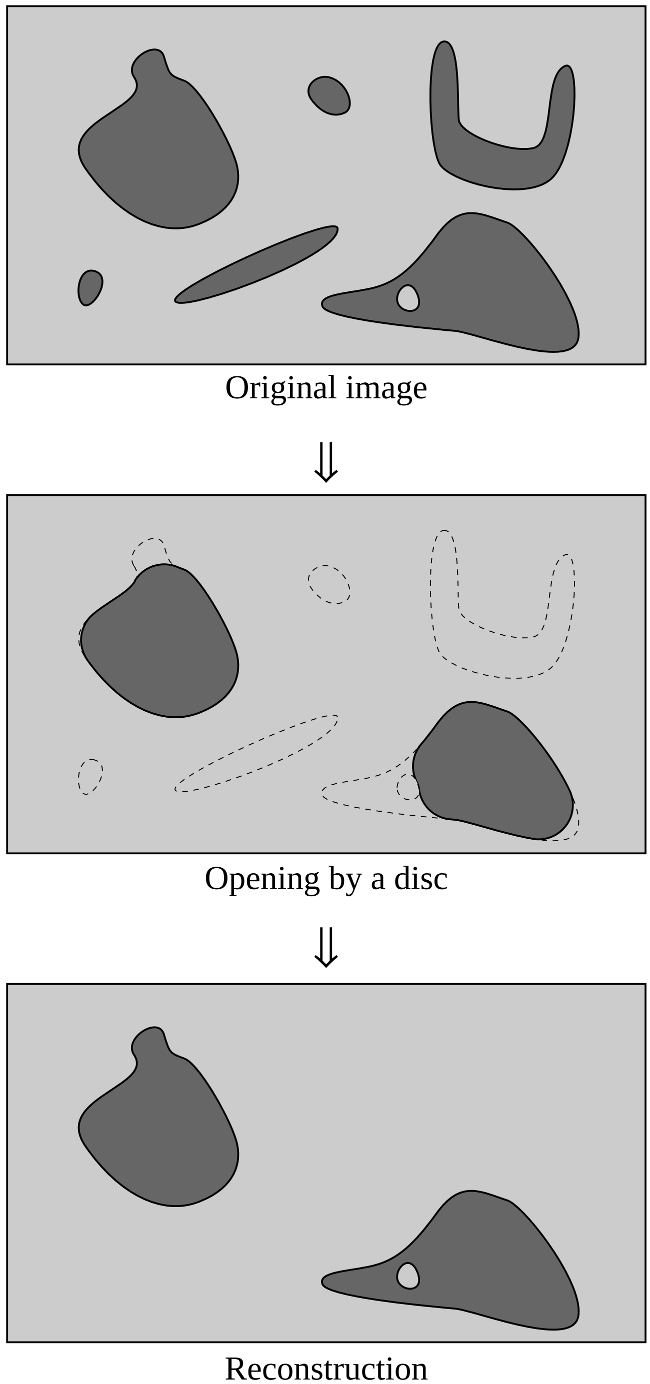


Figure 2.6: Algebraic opening which is defined as a morphological opening by a disc followed by a reconstruction

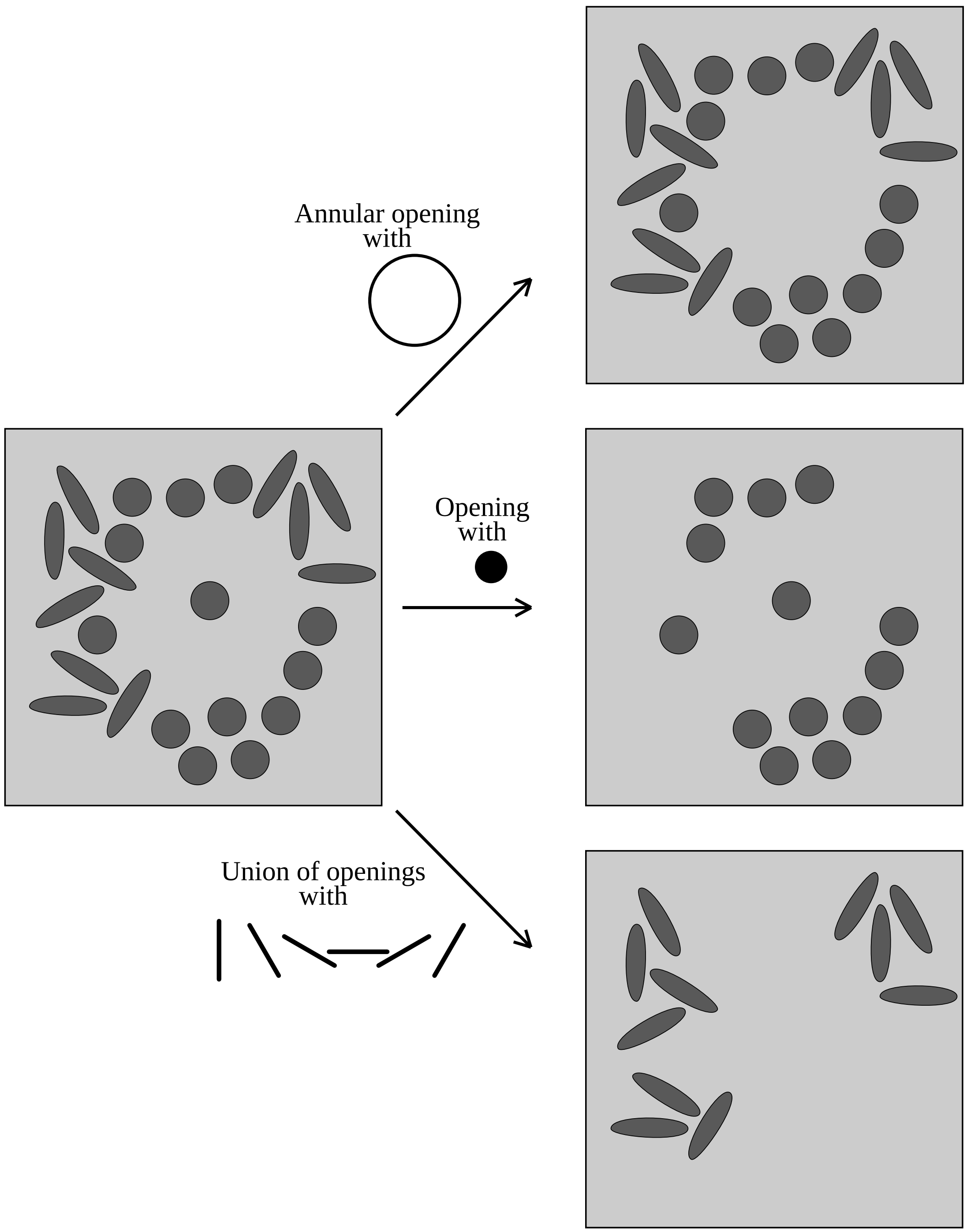


Figure 2.7: Annular opening γ versus a classical opening by a disc and a union of openings by segments.

2.5 Size distributions

Size distributions (also called *granulometries)* deal with families of openings or closings that are parametrized by a positive number (the size) [11]. More precisely, we have the following:

Definition 2.7 *A family* (γ_λ) *of mappings from* $\mathcal{T}$ *into* $\mathcal{T}$, *depending on a positive parameter* λ *is a size distribution when*

(i) $\forall\lambda > 0, \quad \gamma_\lambda$ *is an opening,*

and when one of the three equivalent conditions (ii) to (iv) is fulfilled:

(ii) $\lambda, \mu > 0 \implies \gamma_\lambda \gamma_\mu = \gamma_\mu \gamma_\lambda = \gamma_{\sup(\lambda,\mu)}$

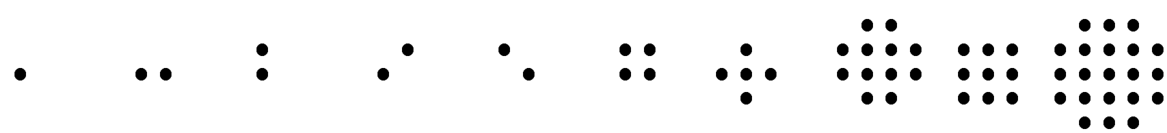
(iii) $\lambda \geq \mu > 0 \implies \gamma_\lambda \leq \gamma_\mu$

(iv) $\lambda \geq \mu > 0 \implies B_\lambda \subseteq B_\mu$.

Similarly, we introduce also *anti-size distributions* as the families of closings (φ_λ), whose dual openings build size distributions.

These conditions are called Matheron's axioms for sized distributions [12, page 192]. It is easy to verify that these conditions are satisfied by every process that common sense would designate as a size distribution.

As an example, consider the following structuring elements:



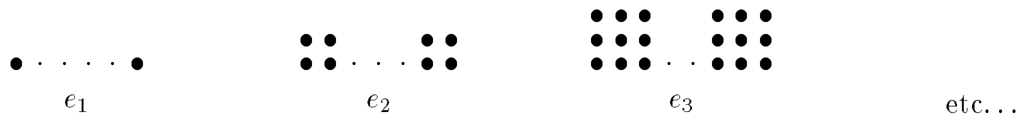
e_0 e_1 e_2 e_3 e_4 e_5 e_6 e_7 e_8 e_9

Then, among the various sequences

γ_0	γ_0	γ_0	γ_0	γ_0	γ_0
$\gamma_1 \vee \gamma_2$	$\gamma_1 \vee \gamma_2 \vee \gamma_3 \vee \gamma_4$	$\gamma_1 \vee \gamma_2$	$\gamma_3 \vee \gamma_4$	$\gamma_1 \vee \gamma_2$	γ_5
γ_5	γ_6	γ_5	γ_5	γ_5	γ_7
γ_7	γ_7	γ_7	γ_8	γ_6	γ_8
γ_9	γ_9	γ_9	γ_9	γ_8	γ_9

the four former ones lead to size distributions, but not the last two ones.

N.B: Remark that the convexity of the structuring elements is not a necessity. For instance, the sequence



induces a size distribution. However, in the Euclidean space $\mathbb{R}^n$, a family $(B_\lambda)_{\lambda\geq 0}$ of structuring elements generates a size distribution $(\gamma_\lambda)_{\lambda\geq 0}$ which is compatible with the magnification, i.e.

$$\forall\lambda \geq 0, \forall X \subset \mathbb{R}^n, \quad \gamma_\lambda(X) = \lambda\gamma_1(X/\lambda), \tag{2.7}$$

if and only of the B_λ's are the homotethics of a compact **convex** set B. The signification of rel. (2.7) is clear: it just means that γ_λ acts on λX just as γ_1 does on X. Such a property, which is always satisfied for convolution products, may not exist for morphological filters. However, in the two important cases of the size distributions and of the alternating sequential filters, we easily obtain it.

Chapter 3

Morphological filters

3.1 The lattice of the increasing mappings

This chapter as well as the next one constitute an overview of the theory of morphological filtering, due to G. Matheron [19, chapter 6]. The lattice examples introduced in chapter 1 concerned the scenes under study. We will now consider classes of *operations* working on these objects. Let ψ be such an operator, i.e. be a mapping from a complete lattice $\mathcal{T}$ into itself. We assume that ψ is increasing, i.e. that it preserves the ordering relation of $\mathcal{T}$:

$$\forall A, A' \in \mathcal{T}, \qquad A \geq A' \Longrightarrow \psi(A) \geq \psi(A'). \tag{3.1}$$

The set $\mathcal{T}'$ of the increasing mappings on the complete lattice $\mathcal{T}$ satisfies the following properties:

1. $\mathcal{T}'$ is a semi-group for the composition product $\circ$, with a unit element, namely the identity mapping I ($\forall A \in \mathcal{T}, I(A) = A$).
2. $\mathcal{T}'$ is a complete lattice for the ordering relation:

$$f \geq g \iff \forall A \in \mathcal{T}, f(A) \geq g(A),$$

since the following identities

$$(\vee_{\mathcal{T}'} f_i)(A) = (\vee_{\mathcal{T}} f_i) \qquad \text{and} \qquad (\wedge_{\mathcal{T}'} f_i)(A) = (\wedge_{\mathcal{T}} f_i)$$

generate a supremum and an infimum in the set $\mathcal{T}'$.

The two basic structures of the semi-group and of the lattice interact with each other, and we have, for all f, g, h and (f_i) in $\mathcal{T}'$:

$$(\vee f_i) \circ g = \vee(f_i \circ g) \quad ; \quad g \circ (\vee f_i) \geq \vee(g \circ f_i) \tag{3.2}$$

$$(\wedge f_i) \circ g = \wedge(f_i \circ g) \quad ; \quad g \circ (\wedge f_i) \leq \wedge(g \circ f_i) \tag{3.3}$$

and

$$f \geq g \Longrightarrow \begin{cases} f \circ h & \geq & g \circ h \\ h \circ f & \geq & h \circ g \end{cases}$$

In the following, the two classes of the *overfilters* (i.e. the mappings $f \in \mathcal{T}'$ such that $f \circ f \geq f$) and of the *underfilters* play a major role. Indeed:

Theorem 3.1 *the class of the underfilters (resp. overfilters) is closed under $\wedge$ (resp. $\vee$) and under self-composition.*

For example, let $(f_j)_{j\in J}$ be a family of underfilters. From (3.1) and (3.3), we get:

$$(\vee_{j\in J} f_j)\circ(\vee_{j\in J} f_j) = \vee_{i\in J}(f_i \circ \vee_{j\in J} f_j) \leq \vee_{i\in J}(f_i \circ f_i) \leq \vee_{i\in J} f_i,$$

so that $\vee_{j\in J} f_j$ is an underfilter. Moreover, given an underfilter f, $ff \leq f$ implies, by growth, that $ff \circ ff \leq ff$, so that the self-composition ff is an underfilter.

3.2 Morphological filters

Following G. Matheron and J. Serra [19, chapters 5–6], we define the notion of a **morphological filter** as follows [1]:

Definition 3.2 *The elements of $\mathcal{T}'$ which are both underfilters and overfilters are called (morphological)* **filters.**

In other words, the morphological filters are the transformations acting on the scenes under study (i.e. the lattice $\mathcal{T}$) and which are **increasing** and **idempotent.** We shall denote by $\mathcal{V}$ the class of the filters, with $\mathcal{V} \subseteq \mathcal{T}'$. Remark that the class $\mathcal{V}$ is not closed either under $\vee$, or $\wedge$, or under composition (a counter-example, based on openings, has been exhibited in § 2.2).

This apparent drawback suggests us to investigate more accurately the possible connections of class $\mathcal{V}$ with the composition product and with extrema. Can we find, for example, pairs (f, g) of filters such that $f \circ g$, $g \circ f$, $f \circ g \circ f$, etc...are surely filters (composition problem)? Can we keep the usual ordering relation in $\mathcal{V}$ and equip $\mathcal{V}$ with **new** sup and inf, such that it turns out to become a complete lattice (extrema problem)? These two sorts of questions will build the subject of the next two sections.

3.3 Composition of morphological filters

With any increasing mapping $\psi : \mathcal{T} \longrightarrow \mathcal{T}$, associate:

1. the **image domain** $\psi(\mathcal{T})$, i.e. the set of the transforms by ψ:

$$\psi(\mathcal{T}) = \{\psi(A), A \in \mathcal{T}\},$$

2. the **invariance domain** $\mathcal{B}_\psi$, i.e. the class of those $B \in \mathcal{T}$ which are left unchanged under ψ:

$$\mathcal{B}_\psi = \{B \in \mathcal{T}, \psi(B) = B\}.$$

When ψ is a filter, $\mathcal{B}_\psi$ is often called the *root* of ψ in literature.

[1] In literature, the term “filter” may also be associated with growth only [9, 10], and can even be a synonymous with mapping [26]

We always have $\mathcal{B}_\psi \subseteq \psi(\mathcal{T})$, an inclusion which becomes an equality

$$\mathcal{B}_\psi = \psi(\mathcal{T})$$

if and only if ψ is **idempotent.** This preliminary remark leads to the following two criteria:

Criterion 3.3 *For any mappings f, g from $\mathcal{T}$ into itself,*

$$fg = g \Longleftrightarrow g(\mathcal{T}) \subseteq \mathcal{B}_f.$$

In particular, when g is idempotent:

$$fg = g \Longleftrightarrow \mathcal{B}_g \subseteq \mathcal{B}_f. \tag{3.4}$$

Criterion 3.4 *Two mappings f and g from $\mathcal{T}$ into itself are idempotent and admit the same invariance domain $\mathcal{B}_f = \mathcal{B}_g$ if and only if:*

$$fg = g \qquad \text{and} \qquad gf = f. \tag{3.5}$$

Proofs: criterion 3.3 is obvious. Now, if rel. (3.5) is satisfied, then $ff = f \circ gf = gf = f$, i.e. f, and similarly g, are idempotent. Hence, from (3.4), $\mathcal{B}_f \subseteq \mathcal{B}_g$ and $\mathcal{B}_g \subseteq \mathcal{B}_f$. Conversely, when f and g, idempotent, have the same invariance domain, rel. (3.5) is nothing but (3.4). □

In these two criteria, the ordering $\leq$ does not intervene. From now on, we shall only consider the **increasing mappings** ψ, i.e. $\psi \in \mathcal{T}'$. For any filter ψ, the class of the filters ψ' that have the same invariance domain $\mathcal{B}_\psi$ as ψ will be denoted $\mathcal{I}d(\psi)$. The following theorem is the key result concerning the composition of filters:

Theorem 3.5 (structural theorem) *Let f and g be two filters on $\mathcal{T}$ such that $f \geq g$. Then:*

(i) $f \geq fgf \geq gf \vee fg \geq gf \wedge fg \geq gfg \geq g$,

(ii) gf, fg, fgf *and* gfg *are filters, and* $fgf \in \mathcal{I}d(fg)$, $gfg \in \mathcal{I}d(gf)$,

(iii) fgf *is the smallest filter greater than* $gf \vee fg$ *and* gfg *is the greatest filter smaller than* $gf \wedge fg$,

(iv) the following equivalences hold:

$$\begin{array}{lllll} \mathcal{B}_{fg} = \mathcal{B}_{gf} & \Longleftrightarrow & \mathcal{B}_{fg} = \mathcal{B}_f \cap \mathcal{B}_g & \Longleftrightarrow & \mathcal{B}_{gf} = \mathcal{B}_f \cap \mathcal{B}_g \\ & \Longleftrightarrow & fgf = gf & \Longleftrightarrow & gfg = fg \\ & \Longleftrightarrow & gf \geq fg. & & \end{array}$$

proof: The inequalities *(i)* are obvious.

From the relationships

$$fg = fffg \geq fgfg \geq fggg = fg,$$

we conclude that fg is a filter. By the dual inequalities, gf is also a filter. Now, we have

$$\begin{aligned} fgf \circ fg &= fg(ff)g = fgfg = fg, \\ fg \circ fgf &= fgfg \circ f = (fg \circ fg)f = fgf, \end{aligned}$$

and thus, $fgf \in \mathcal{I}d(fg)$, by criterion 3.4. In the same way, we find that $fgf \in \mathcal{I}d(gf)$, so that *(ii)* is proved.

Now, fgf is a filter (by *(ii)*) and $fgf \geq gf \vee fg$ (by *(i)*). Let ψ be a filter such that $\psi \geq fg$ and $\psi \geq gf$. It follows that $\psi = \psi\psi \geq fggf = fgf$. Thus, fgf is the smallest filtering upper bound of fg and gf. Hence *(iii)* is proved.

By criterion 3.4, we have $\mathcal{B}_{fg} = \mathcal{B}_{gf}$ if and only if

$$fg \circ gf = fgf = gf \quad \text{and} \quad gf \circ fg = gfg = fg.$$

These relations actually imply one another. For instance, $fgf = gf$ implies $fgf \circ g = gfgf$, i.e. $fg = gfg$. By *(iii)*, these relations are equivalent to $gf \geq fg$.

The inclusions

$$\mathcal{B}_f \cap \mathcal{B}_g \subseteq \mathcal{B}_{fg} \subseteq \mathcal{B}_f$$

always hold, so that $\mathcal{B}_{fg} = \mathcal{B}_f \cap \mathcal{B}_g$ if and only if $\mathcal{B}_{fg} \subseteq \mathcal{B}_g$, i.e, by criterion 3.3, if and only if $gfg = fg$. This completes the proof. □

Examples:

1. Start from an arbitrary opening γ and an arbitrary closing ϕ. Since

$$\gamma \leq I \leq \phi,$$

by theorem 3.5, $\gamma\phi$, $\phi\gamma$, $\gamma\phi\gamma$ and $\phi\gamma\phi$ are filters. The composition products of ϕ by γ, then by ϕ, etc... generates the oscillating sequence

$$\phi \longrightarrow \gamma\phi \longrightarrow \phi\gamma\phi \longrightarrow \gamma\phi \longrightarrow \dots$$

Remark that when $\gamma\phi \geq \phi\gamma$ (which is generally not the case), we have $\gamma\phi = \phi\gamma\phi$ and the oscillations are stopped after the first step.

2. There is a more particular example, which illustrates point *(iv)* of the theorem. In $\mathcal{P}(\mathbb{R}^n)$ (or $\mathcal{P}(\mathbb{Z}^n)$, or $\mathcal{F}(\mathbb{R}^n, \mathbb{R})$, or $\mathcal{F}(\mathbb{Z}^n, \mathbb{Z})$), consider the morphological opening γ_l by a segment of length l in the horizontal direction. For a given X, $\gamma_l(X)$ is made of horizontal segments of length $\geq l$. Moreover, closing this set by the dual closing ϕ_l, i.e. determining $\phi_l\gamma_l(X)$ may only suppress intervals between two such segments, hence increase the length of the horizontal intercepts; Therefore, $\gamma_l\phi_l\gamma_l(X) = \phi_l\gamma_l(X)$, and by the theorem, $\phi_l\gamma_l \leq \gamma_l\phi_l$.

3.4 The lattice of the filters

We now go back to the structure of the set $\mathcal{V}$ of the filters acting on lattice $\mathcal{T}$. Clearly, if (ψ_i) is a family of elements of $\mathcal{V}$, then $\vee\psi_i$ is an overfilter,

$$(\vee_i\psi_i)(\vee_i\psi_i) = \vee_i(\psi_i(\vee_j\psi_j)) \geq \vee_i(\psi_i \circ \psi_i) = \vee_i\psi_i,$$

and similarly, $\wedge\psi_i$ is an underfilter. Since $\vee\psi_i$ is an overfilter, the class $\mathcal{C}$ closed under $\vee$ and self-composition generated by $\vee\psi_i$ only comprises overfilters (theorem 3.1). It admits a largest element f, for $\mathcal{T}'$ is a complete lattice. But $f \circ f$ also belongs to $\mathcal{C}$ (closure under self-composition), hence $f \circ f \leq f$, i.e. f is a **filter.**

Consider now a filter ψ' larger than $\vee\psi_i$. The class $\mathcal{C}'$ of the overfilters smaller than ψ' is, in turn, closed under $\vee$ and self-composition, and contains the previous class $\mathcal{C}$. Hence, $f \leq \psi'$, i.e. f turns out to be the smallest filter which majorates the ψ_i's. By duality, we have a similar result for the inf, and we may state:

Theorem 3.6 *The set $\mathcal{V}$ of the filters on $\mathcal{T}$ is a complete lattice. For any family (ψ_i) of filters on $\mathcal{T}$, the smallest filter greater than $\vee\psi_i$ is the largest element f of the class closed under $\vee$ and self-composition generated by (ψ_i). Dual result for the largest filter smaller than $\wedge\psi_i$.*

In particular, when $\mathcal{T}$ is finite, we always have, for a large enough n:

$$f = (\vee\psi_i)^n, \tag{3.6}$$

a result which provides the algorithm for computing f in practice.

Examples:

1. Lattice of the openings: take the class $\mathcal{V}' \subseteq \mathcal{V}$ of the openings on $\mathcal{T}$. We have seen that for every family (γ_i) in $\mathcal{V}'$, $\vee\gamma_i$ is still an opening (theorem 2.3). Moreover, from theorem 3.6, there exists a largest filter g which is smaller than all the γ_i's. g being obviously anti-extensive, $\mathcal{V}'$ is a complete lattice.

2. Start from an arbitrary increasing mapping ψ. Then, the extensive mapping $I \vee \psi$ is an overfilter and the proof of the theorem shows that there exists a smaller filter $\hat{\psi}$—hence a smaller **closing**—that majorates ψ. In the finite case, we find again theorem 2.6.

Chapter 4

$\vee$- and $\wedge$-filters, $\gamma\phi$ and $\phi\gamma$, strong filters

4.1 Introduction

We say that a mapping $f : \mathcal{T} \longrightarrow \mathcal{T}$ is a $\vee$-mapping when

$$f = f \circ (I \vee f) \tag{4.1}$$

and a $\wedge$-mapping when

$$f = f \circ (I \wedge f). \tag{4.2}$$

Basically, this property is something new and independent from the two axioms which build the definition of the morphological filters. If now f is increasing and satisfies rel. (4.1), we shall call it a $\vee$-underfilter. Indeed, any $\vee$-underfilter is an underfilter and similarly, any $\wedge$-overfilter is an overfilter. If f is, for instance, a $\vee$-underfilter, then we have:

$$f = f \circ (I \vee f) \geq f \vee ff \geq f.$$

Thus, $f = f \vee ff$ is an underfilter.

A filter which satisfies rel. (4.1) (resp rel. (4.2)) will be called a $\vee$-filter (resp. a $\wedge$-filter). When it satisfies both rel. (4.1) and (4.2) it will be said to be a **strong filter.** The geometrical interpretation of $\vee$-filtering and of $\wedge$-filtering are very easy. Indeed, ψ is a $\vee$-filter if and only if, for any $A \in \mathcal{T}$, every B between A and $A \vee \psi(A)$ has the same transform as A itself (see Fig. 4.1), i.e.

$$\psi \ \vee -\text{filter} \qquad \forall A \in \mathcal{T}, (A \leq B \leq A \vee \psi(A) \Longrightarrow \psi(B) = \psi(A)).$$

Similarly, we have

$$\psi \ \wedge -\text{filter} \qquad \forall A \in \mathcal{T}, (A \wedge \psi(A) \leq B \leq A \Longrightarrow \psi(B) = \psi(A)).$$

The following result corresponds to the theorem 3.1 of the general case—and is proved in the same way:

Theorem 4.1 *the class of the $\vee$-underfilters (resp. $\wedge$-overfilters) is closed under $\vee$ (resp. under $\wedge$) and self-composition.*

(Note the chiasma; it is the sup and not the inf of $\wedge$-overfilters which is still an $\wedge$-overfilter.) This theorem suggests us to approach first the properties of self-composition, and then that of sup and inf, just as we did in the general case.

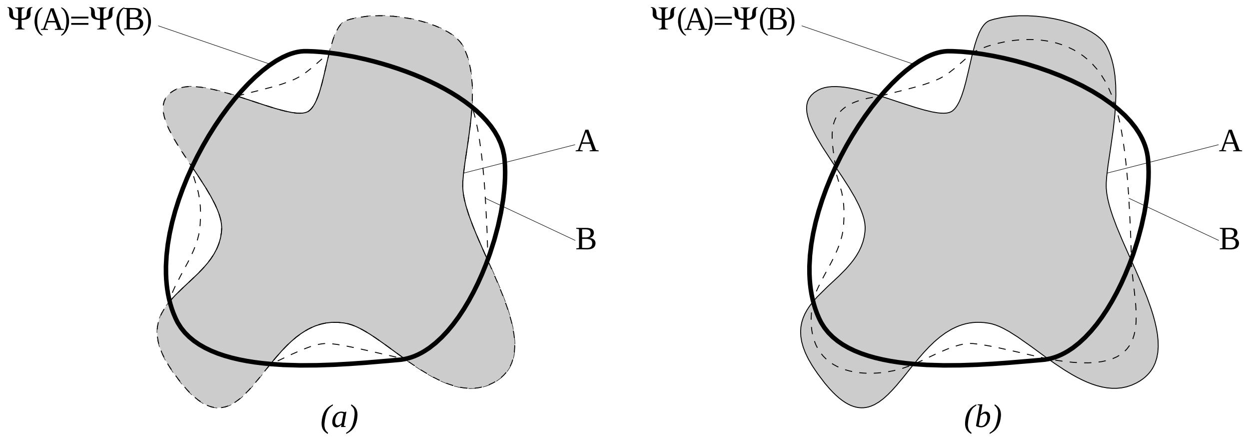


Figure 4.1: An example of a ∨-filter (a) and of a strong filter (b).

4.2 Composition of ∨- and ∧-filters

Theorem 4.2 *Let f and g be two filters on $\mathcal{T}$ and $f \geq g$. Then*

(i) if f is a ∨-filter, gf and fgf are ∨-filters,

(ii) if f is a ∧-filter, fg and gfg are ∧-filters,

(iii) if gf is a ∧-filter, fgf is a ∧-filter,

(iv) if fg is a ∨-filter, gfg is a ∨-filter.

proof: Easy [19, page 119]. □

Example:

If γ is an *opening* and ϕ a *closing*, we have $\gamma \leq I \leq \phi$, and then

- γ and ϕ are strong filters,
- $\gamma\phi$ and $\phi\gamma\phi$ are ∨-filters,
- $\phi\gamma$ and $\gamma\phi\gamma$ are ∧-filters.

Moreover, if $\gamma\phi$ is a ∧-filter, and thus a strong filter, $\phi\gamma\phi$ is a strong filter. In the same way, if $\phi\gamma$ is a strong filter, then $\gamma\phi\gamma$ is a strong filter.

4.3 The two envelopes $\psi\hat{\psi}$ and $\psi\check{\psi}$

In § 3.4, we have associated with each $\psi \in \mathcal{T}'$ the largest opening $\check{\psi}$ that minorates ψ and the smallest closing $\hat{\psi}$ which majorates ψ. These two primitives play a central role in the ∨- and ∧-characterizations, as is shown by the following theorem:

Theorem 4.3 *An increasing mapping $\psi : \mathcal{T} \longrightarrow \mathcal{T}$ is a $\vee$-underfilter (resp. a $\wedge$-overfilter) if and only if $\psi = \psi\hat{\psi}$ (resp. $\psi = \psi\check{\psi}$).*

proof: If $\psi = \psi(I \vee \psi)$, then

$$(I \vee \psi)(I \vee \psi) = I \vee \psi \vee \psi(I \vee \psi) = I \vee \psi \vee \psi = I \vee \psi.$$

The mapping $I \vee \psi$, which is idempotent and which majorates I, is nothing but $\hat{\psi}$, and $\psi(I \vee \psi) = \psi\hat{\psi}$.

Conversely, start from

$$\hat{\psi} \leq \hat{\psi}(I \vee \psi) \leq \hat{\psi}(I \vee \hat{\psi}) = \hat{\psi},$$

which implies $\hat{\psi} = \hat{\psi}(I \vee \psi)$. Now, if $\psi = \psi\hat{\psi}$, then
$\psi = \psi\hat{\psi} = \psi\hat{\psi}(I \vee \psi) = \psi(I \vee \psi)$. □

Corollary 4.1 *If $\psi \in \mathcal{T}'$ is an overfilter (resp. an underfilter), then $\psi\hat{\psi}$ is the smallest $\vee$-filter which majorates ψ (resp. the largest $\wedge$-filter which minorates ψ).*

proof: If ψ is an overfilter, then

$$\psi\hat{\psi} \leq \psi\psi\hat{\psi} \leq \psi\hat{\psi}\psi\hat{\psi} \leq \psi\hat{\psi}\hat{\psi}\hat{\psi} = \psi\hat{\psi}$$

and $\psi\hat{\psi}$ is a filter. It is also a $\vee$-filter, since

$$\psi\hat{\psi} \leq \psi\hat{\psi}(I \vee \psi\hat{\psi}) \leq \psi\hat{\psi}(I \vee \hat{\psi}\hat{\psi}) = \psi\hat{\psi},$$

which implies $\psi\hat{\psi}(I \vee \psi\hat{\psi}) = \psi\hat{\psi}$. Finally, if ψ' is a $\vee$-filter, with $\psi' \geq \psi$, then $\psi' = \psi'\hat{\psi}' \geq \psi$ implies $\hat{\psi}' \geq \hat{\psi}$, hence $\psi' = \psi'\hat{\psi}' \geq \psi\hat{\psi}$. □

Corollary 4.2 *When $\mathcal{T}$ is a finite lattice, then*

$$\psi\hat{\psi} = \psi(I \vee \psi)^n \tag{4.3}$$

for n large enough.

Indeed, corollary 4.1 associates two envelopes with each filter ψ, so that ψ is finally surrounded by four extremum filters as follows:

$$\check{\psi} \leq \psi\check{\psi} \leq \psi \leq \psi\hat{\psi} \leq \hat{\psi}. \tag{4.4}$$

We have seen that the product $\gamma\phi$ of any closing followed by any opening was a $\vee$-filter. We will prove now that the converse is true, so that the $\vee$-property **characterizes** the class of the filters of the type $\gamma\phi$.

Theorem 4.4 *A mapping $\psi \in \mathcal{T}'$ is a $\vee$-filter (resp. a $\wedge$-filter) if and only if there exist an opening γ and a closing ϕ such that $\psi = \gamma\phi$ (resp. $\psi = \phi\gamma$).*

proof: Assume that ψ is a $\vee$-filter and consider its invariance domain $\mathcal{B}$. Denote by $\underset{\sim}{\mathcal{B}}$ the class closed for the sup which is generated by $\mathcal{B}$, and by $\underset{\sim}{I}$ the associated opening. Clearly, we have $\psi \geq \underset{\sim}{I}$. Moreover, according to criterion 3.4, $\mathcal{B} \subseteq \underset{\sim}{\mathcal{B}}$ implies $\psi = \underset{\sim}{I}\,\psi$ and (theorem 4.3) $\psi = \underset{\sim}{I}\,\psi\hat{\psi}$. Thus, we may write:

$$\psi = \underset{\sim}{I}\,\psi = \underset{\sim}{I}\,\psi\hat{\psi} \begin{cases} \geq \underset{\sim}{I}\,\hat{\psi} & \text{for} \quad \psi \geq \underset{\sim}{I}, \\ \leq \underset{\sim}{I}\,\hat{\psi} & \text{for} \quad \psi \leq \hat{\psi}. \end{cases}$$

Hence, $\psi = \underset{\sim}{I}\,\hat{\psi}$, i.e. the composition product of a closing by an opening. □

Remark: the above decomposition is not unique. We also have, for a $\vee$-filter ψ, $\psi = \check{\psi}\hat{\psi}$.

Example: the rank operators

We will illustrate the important theorem 4.3 by considering some properties of the rank operators [16, 7, 17]. Let f be a function from $\mathbb{Z}^n$ into $\mathbb{Z}$ and let $B \subset \mathbb{Z}^n$ be a finite and symmetrical set of p points implanted at the origin. B is the (moving) window in which the ranking operation will be implemented. The transform R_k of rank k of f at point $x \in \mathbb{Z}^n$ is obtained by ordering the family $(f(y))_{y \in B_x}$ with decreasing values (for example), and by replacing $f(x)$ by the k-th value of the sequence that is thus constructed ($1 \leq k \leq p$). For $k = p$ and $k = 1$, this leads to Minkowski addition and subtraction. When $k = \frac{1}{2}(p+1)$ and p is an odd number, the resulting operation is sometimes called *median filtering,* [26] in literature.

The rank operator R_k of rank k is increasing, since it can be decomposed into the sup of the t-erosions ε_i by all the $B_i \subseteq B$ which possess k points:

$$R_k(f) = \vee\{\varepsilon_i(f), B_i \subseteq B, \mathrm{Card}(B_i) = k\}. \tag{4.5}$$

Now, $\psi_i = \delta_B \varepsilon_i$ is a $\wedge$-overfilter. Indeed:

$$\psi_i \gamma_i = \delta_B \varepsilon_i \delta_i \varepsilon_i \begin{cases} \geq \delta_B \varepsilon_i = \psi_i & \text{for} \quad \varepsilon_i \delta_i \geq I, \\ \leq \delta_B \varepsilon_i = \psi_i & \text{for} \quad \delta_i \varepsilon_i \leq I, \end{cases}$$

and $\psi_i \gamma_i = \psi_i$. On the other hand, $\gamma_i \leq \psi_i$ implies $\gamma_i \leq I \wedge \psi_i \leq I$, hence $\gamma_i = \gamma_i(I \wedge \psi_i)$ and finally, $\psi_i = \psi_i(I \wedge \psi_i)$. Rel. (4.5) shows that $\delta_B R_k = \vee \psi_i$ is still a $\wedge$-overfilter (theorem 4.1), and by application of theorem 4.3, $I \wedge \delta_B R_k$ is an opening (called Ronse opening). In particular, for $k = 1$, one finds the morphological opening by B, and for $k = p$, the identity I. The other openings increase with k.

As another application of the results presented in this section, one can refer to an interesting study of F. Meyer [15], where the author directly transcribes into practice and into algorithms the above theorems. Some of the filters which are thus brought to the fore directly stem from classical median filters [26]. Besides, in two papers by P. Maragos [9, 10] which are among the most interesting ones in the recent literature on morphological filters, one can find a thorough study on the relations between morphological filters and non-morphological ones, namely median filters, rank filters and stack filters...

4.4 The lattice of the strong filters

Starting from an $\wedge$-overfilter f', the first corollary of theorem 4.3 ensures that $f = f'\hat{f}'$ is an $\vee$-filter. It would be excellent if it could also keep the $\wedge$-overfiltering property of its primitive f'. In this case, we would have found the key for producing **strong** filters. The answer will actually be positive in the case of modular lattices $\mathcal{T}$, i.e. such that

$$\forall A, B, C \in \mathcal{T}, \qquad B \geq A \implies (A \vee C) \wedge B = A \vee (B \wedge C)$$

(Except the partition lattice, all the lattices used as models in morphology are modular.). Then, we have the following lemma:

Lemma 4.5 *When $\mathcal{T}$ is modular, then*

1. *$f \circ (I \wedge f)$ is a $\vee$-filter for any $\vee$-filter f,*

2. $g \circ (I \vee g)$ is a $\wedge$-filter for any $\wedge$-filter g.

proof: Easy. Refer to [19, page 124]. □

Theorem 4.6 *When the lattice $\mathcal{T}$ is modular, if f' is a $\wedge$-overfilter (resp. an $\wedge$-underfilter), then $f = f'\hat{f}'$ (resp. $g = g'\check{g}'$) is a strong filter.*

proof: Let f' be a $\wedge$-overfilter and $f = f'\hat{f}'$. We have

$$f' = f'(I \wedge f') \leq f(I \wedge f) \leq f.$$

Now, from the lemma, $f(I \wedge f)$ is a $\vee$-filter. Since it also majorates f', it is larger than the smallest $\vee$-filter which majorates f', namely f. Hence, $f = f(I \wedge f)$ is strong. □

Corollary 4.3 *When $\mathcal{T}$ is modular, the class of the strong filters on $\mathcal{T}$ is a complete lattice based on the usual ordering. The supremum of a family (ψ_i) of strong filters is $f'\hat{f}'$, with $f' = \vee\psi_i$, and the infimum is given by $g'\check{g}'$, with $g' = \wedge\psi_i$.*

Examples:

1. Theorem 4.6 opens the way for the construction of as many strong filters as we wish, by iterations. It suffices to start from an arbitrary opening γ and an arbitrary closing ϕ: when lattice $\mathcal{T}$ is finite, there exist two integers n and p such that both mappings

$$\psi = \phi\gamma(I \vee \phi\gamma)^n \qquad \text{and} \qquad \psi' = \gamma\phi(I \wedge \gamma\phi)^n$$

   are strong filters.

2. In some cases, it is not necessary to perform iterations. It is the case, for instance, of the morphological opening γ_l and the closing ϕ_l with respect to a segment l (see § 3.3). The points which change from 0 to 1 in $I \vee \phi_l\gamma_l$ are uniquely those which were initially modified from 1 to 0 by γ_l, and preserved as 0's by ϕ_l(See the example of Fig. 4.2. When γ_l acts on $I \vee \phi_l\gamma_l$, we then recover $\phi_l\gamma_l$. Hence $\phi_l\gamma_l$ is strong. However, it is not self-dual, since by theorem 3.5 *(iv)*, we have $\gamma_l\phi_l \geq \phi_\lambda\gamma_\lambda$.

4.5 Conclusion

In the framework of morphological filtering, the $\vee$- and $\wedge$-properties are weaker substitutes for extensivity and anti-extensivity (closings and openings are strong filters, but the converse is false). Such a weaker version allows us to combine both properties in filters that are not trivial, whereas the only strong filter to be extensive and anti-extensive at the same time is the identity mapping I. We will see in the next chapter how this advantage can be used to produce self-dual filters.

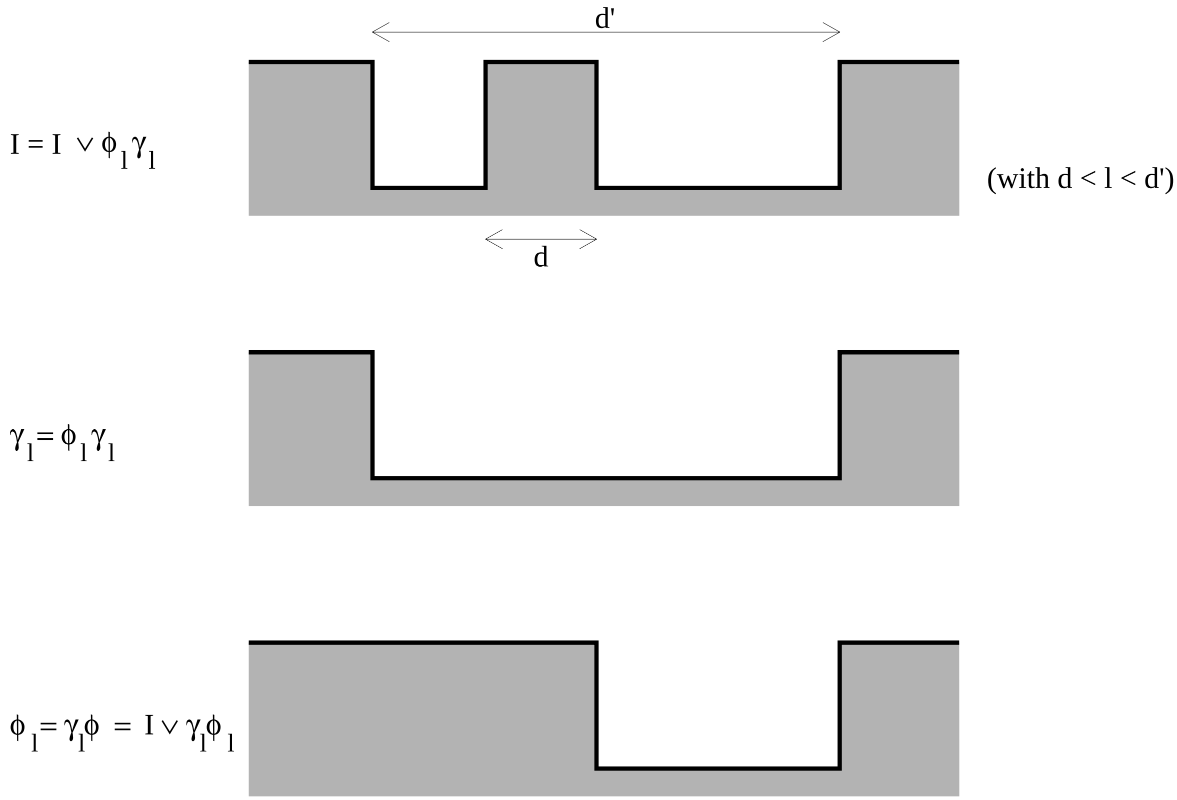


Figure 4.2: A case where it is not necessary to perform iterations: the opening γ_l and the closing ϕ_l with respect to a segment l.

Chapter 5

Alternating Sequential Filters

This chapter is devoted to the presentation of a particular class of filters, namely the *alternating sequential filters (ASF)*. It provides a simplified version of the results proved in [19, chapter 10]. The theory developed in [19] requires *monotonous sequential continuity* for covering both continuous and discrete cases. For the sake of simplicity, we will restrict ourselves to the latter case (i.e. $\mathcal{P}(\mathbb{Z}^n)$, $\mathcal{F}(\mathbb{Z}^n, \mathbb{Z})$ or planar graphs), which allows us to avoid monotonous topology.

5.1 Introduction

The appearance of the alternating sequential filters in the world of mathematical morphology is due to an experimental work of Sternberg [22]. His study consisted in taking a polyhedric form (namely a cuboctahedron, see § 1.6), altering it by the addition of a largely varying white noise, and then trying to clean the resulting image X (see [19, page 204]). To attain this goal, X was first filtered by a small closing ϕ_1, followed by a small opening γ_1, then by a slightly larger closing ϕ_2 followed by a slightly larger opening γ_2, etc... The final operator produced by this succession of openings and closings was

$$M = (\gamma_i \phi_i) \circ \ldots \circ (\gamma_2 \phi_2) \circ (\gamma_1 \phi_1).$$

The family (ϕ_i) that was used in this example consisted in morphological closings by homothetic structuring elements, whereas (γ_i) was the dual family of openings. After this experiment, a certain number of questions arose: is the operator M a filter? To what extend does it depend on the totality of the sequence of parameters 1, 2,..., i? Is it essential to use a size distribution (γ_i) and its dual (ϕ_i)? Is the product of these operators an operator of the same type?...

In this section, a more formal approach of the above operators is proposed. The ASF's of type M are defined, as well as other classes of ASF's, namely that of type N, R and S. A number of properties of these operators, dealing with composition and order relation, are then reviewed. Transposed ASF's and symmetrical ones are also presented. The end of the section is concerned with more practical problems, such as computation time and the use of these filters for concrete applications.

5.2 Definitions

In the following, $\mathcal{T}$ denotes a discrete lattice such as $\mathcal{P}(\mathbb{Z}^n)$ or $\mathcal{F}(\mathbb{Z}^n, \mathbb{Z})$. We also consider two families $(\gamma_i)_{i\geq 1}$ and $(\phi_i)_{i\geq 1}$—indexed on $\mathbb{Z}^+/\{0\}$—of operators on $\mathcal{T}$, which are a size distribution and an anti-size distribution respectively, i.e:

$$\forall i \in \mathbb{Z}, i \geq 1, \quad \gamma_i \text{ is an opening and } \phi_i \text{ is a closing.} \tag{5.1}$$

$$\forall i, j \in \mathbb{Z}, 1 \leq i \leq j, \quad \gamma_j \leq \gamma_i \leq I \leq \phi_i \leq \phi_j. \tag{5.2}$$

Moreover, we have seen in definition 2.7 that the inequalities (5.2) are equivalent to the following property:

$$\forall i, j \in \mathbb{Z}, \qquad \gamma_i\gamma_j = \gamma_j\gamma_i = \gamma_{\max(i,j)} \text{ and } \phi_i\phi_j = \phi_j\phi_i = \phi_{\max(i,j)} \tag{5.3}$$

Remark that these two families are chosen **independently** from one another (although they are often taken as dual of each other in practice).

Now, we know (refer to the structural theorem 3.5) that there are four different ways for composing two filters f and g such that $f \leq g$ in order to get new filters. Therefore, we define for all $i \in \mathbb{Z}, i \geq 1$

$$\boxed{\begin{array}{lcllcl} m_i & = & \gamma_i\phi_i & r_i & = & \phi_i\gamma_i\phi_i \\ n_i & = & \phi_i\gamma_i & s_i & = & \gamma_i\phi_i\gamma_i. \end{array}}$$

and we have:

Proposition 5.1 *For all $i \in \mathbb{Z}, i \geq 1$, m_i, n_i, r_i and s_i are filters.*

proof: This is a direct application of the structural theorem 3.5. □

Proposition 5.2 *For $i, j \in \mathbb{Z}$ such that $1 \leq i \leq j$, we have:*

$$m_j m_i \leq m_j \leq m_i m_j \qquad \left\{ \begin{array}{c} r_j r_i \\ r_i r_j \end{array} \right\} \leq r_j \tag{5.4}$$

$$n_i n_j \leq n_j \leq n_j n_i \qquad \left\{ \begin{array}{c} s_j s_i \\ s_i s_j \end{array} \right\} \geq s_j. \tag{5.5}$$

proof: Let us show first that $m_j m_i \leq m_j \leq m_i m_j$. We know that $\gamma_i \leq I$, hence, $\gamma_i\phi_i \leq \phi_i$. Thus, since $\phi_i \leq \phi_j$, the preceding inequality implies

$$\gamma_i\phi_i \leq \phi_j.$$

Now, ϕ_j being increasing, $\phi_j\gamma_i\phi_i \leq \phi_j\phi_j = \phi_j$ (by idempotence of ϕ_j). γ_j is also increasing, hence we finally obtain $\gamma_j\phi_j\gamma_i\phi_i \leq \gamma_j\phi_j$, i.e:

$$m_j m_i \leq m_j,$$

which is the first inequality. Similarly, $I \leq \phi_i$ yields (γ_i increasing) $\gamma_i \leq \gamma_i\phi_i$. Thus, the family (γ_i) being a size distribution, $\gamma_j \leq \gamma_i\phi_i$, which in turn implies $\gamma_j\gamma_j = \gamma_j \leq \gamma_i\phi_i\gamma_j$ (γ_j idempotent). Therefore, $\gamma_j\phi_j \leq \gamma_i\phi_i\gamma_j\phi_j$, i.e:

$$m_j \leq m_i m_j.$$

A similar proof is used for the first part of relation (5.5).

Let us prove now the second part of rel. (5.4), i.e. $r_j r_i \le r_j$ and $r_i r_j \le r_j$. We know that $\phi_i \le \phi_j$. Thus, m_j being increasing, $m_j \phi_i \le m_j \phi_j = \gamma_j \phi_j \phi_j = \gamma_j \phi_j = m_j$. Hence, $m_j \phi_i m_i \le m_j m_i$, which implies that $m_j \phi_i m_i \le m_j$ (previous demonstration). Therefore (ϕ_j increasing), we have $\phi_j m_j \phi_i m_i \le \phi_j m_j$, i.e:

$$r_j r_i \le r_j.$$

Similarly, we know that $n_i n_j \le n_j$, i.e. $n_i \phi_j \gamma_j \le n_j$. But, from (5.3), we know that $\phi_j = \phi_i \phi_j$. Hence, $n_i \phi_i \phi_j \gamma_j \le n_j$, i.e. $r_i n_j \le n_j$. Finally, we get

$$r_i r_j = r_i n_j \phi_j \le n_j \phi_j = r_j.$$

Rel. (5.4) is thus proved. A similar proof is used for the second part of relation (5.5). □

Proposition 5.3 *Let $(i_k)_{1 \le k \le p}$ be p numbers such that*

$$\forall k, i_k \in \mathbb{Z}, 1 \le i_k \le i_1 = i_p.$$

then:

$$\begin{cases} m_{i_p} m_{i_{p-1}} \dots m_{i_2} m_{i_1} & = & m_{i_p} & = & m_{i_1} \\ n_{i_p} n_{i_{p-1}} \dots n_{i_2} n_{i_1} & = & n_{i_p} & = & n_{i_1} \end{cases}$$

proof: Let us prove for instance the first relation. $i_2 \le i_1$, hence, by rel. (5.4), $m_{i_2} m_{i_1} \ge m_{i_1}$. Then, m_{i_3} being increasing, we have $m_{i_3}(m_{i_2} m_{i_1}) \ge m_{i_3} m_{i_1}$, and applying again rel. (5.4) gives, since $i_3 \le i_1$: $m_{i_3}(m_{i_2} m_{i_1}) \ge m_{i_1}$. If we iterate this process, we finally get

$$m_{i_p} m_{i_{p-1}} \dots m_{i_2} m_{i_1} \ge m_{i_1}.$$

Conversely, $i_{p-1} \le i_p$ yields, by rel. (5.4), $m_{i_{p-1}} m_{i_p} \le m_{i_p}$. Then, $m_{i_{p-2}}$ being increasing, we have $m_{i_{p-2}}(m_{i_{p_1}} m_{i_p}) \le m_{i_{p-2}} m_{i_p}$, and applying again rel. (5.4), we obtain ($i_3 \le i_1$): $m_{i_{p-2}}(m_{i_{p-1}} m_{i_p}) \le m_{i_p}$. After iterating this process, we finally get

$$m_{i_p} m_{i_{p-1}} \dots m_{i_2} m_{i_1} \le m_{i_p},$$

which completes the proof of the first relation. Similar proof for the second one. □

We can now give the definition of the alternating sequential filters and prove that they are effectively filters:

Definition 5.4 *For all $i \in \mathbb{Z}, i \ge 1$, the following operators:*

$$\boxed{\begin{array}{lllllll} M_i & = & m_i m_{i-1} \dots m_2 m_1 & \quad & R_i & = & r_i r_{i-1} \dots r_2 r_1 \\ N_i & = & n_i n_{i-1} \dots n_2 n_1 & \quad & S_i & = & s_i s_{i-1} \dots s_2 s_1 \end{array}}$$

are called alternating sequential filters of order i.

Proposition 5.5 *$\forall i \in \mathbb{Z}, i \ge 1$, the operators M_i, N_i, R_i and S_i are filters.*

proof: These operators are increasing as compositions of increasing mappings. Moreover, we have:

$$\begin{aligned} M_i M_i &= (m_i m_{i-1} \dots m_2 m_1)(m_i m_{i-1} \dots m_2 m_1) \\ &= (m_i m_{i-1} \dots m_2 m_1 m_i)(m_{i-1} \dots m_2 m_1) \\ &= m_i (m_{i-1} \dots m_2 m_1) \qquad \text{(by prop. 5.3)} \\ &= M_i. \end{aligned}$$

The idempotence of M_i is thus proved. That of N_i has a similar proof.

As concerns R_i, the proof is a bit more complicated, and we have first to prove the following lemma:

Lemma 5.6 *For all $i \in \mathbb{Z}, i \geq 1$, $R_i = \phi_i M_i$ and $S_i = \gamma_i N_i$.*

proof: This is a straightforward consequence of the decomposition of R_i into elementary openings and closings:

$$\begin{aligned} R_i &= (\phi_i\gamma_i\phi_i)(\phi_{i-1}\gamma_{i-1}\phi_{i-1})\dots(\phi_2\gamma_2\phi_2)(\phi_1\gamma_1\phi_1) \\ &= \phi_i\gamma_i(\phi_i\phi_{i-1})\gamma_{i-1}(\phi_{i-1}\phi_{i-2})\dots(\phi_2\phi_1)\gamma_1\phi_1 \\ &= \phi_i\gamma_i(\phi_i)\gamma_{i-1}(\phi_{i-1})\dots(\phi_2)\gamma_1\phi_1 \\ &= \phi_i \circ m_i m_{i-1}\dots m_2 m_1 \\ &= \phi_i M_i. \end{aligned}$$

In the same way, we show that $S_i = \gamma_i N_i$. □

Let's go back now to the idempotence of R_i.

1. Starting from $I \leq \phi_i$, we get $M_i \leq M_i\phi_i$. Thus, $M_i = M_iM_i \leq M_i\phi_iM_i$ and finally, $R_i = \phi_iM_i \leq \phi_iM_i\phi_iM_i = R_iR_i$.
2. To show the inverse inequality, we have to decompose the operator R_iR_i into simple operations:

$$\begin{aligned} R_iR_i &= \phi_iM_i\phi_iM_i \\ &= \phi_i(\gamma_i\phi_i\gamma_{i-1}\phi_{i-1}\dots\gamma_2\phi_2\gamma_1\phi_1)\phi_i(\gamma_i\phi_iM_{i-1}) \\ &= \phi_i\gamma_i[(\phi_i\gamma_{i-1})(\phi_{i-1}\gamma_{i-2})\dots(\phi_2\gamma_1)](\phi_1\phi_i)\gamma_i(\phi_iM_{i-1}) \\ &= \phi_i\gamma_i[(\phi_i\gamma_{i-1})(\phi_{i-1}\gamma_{i-2})\dots(\phi_2\gamma_1)]\phi_i\gamma_i(\phi_iM_{i-1}) \end{aligned}$$

Consider now the two families (γ'_j) and (ϕ'_j) such that

$$\forall j, \begin{cases} \phi'_j &= \begin{cases} \phi_{j+1} & \text{for } j < i, \\ \phi_j & \text{for } j \geq i. \end{cases} \\ \gamma'_j &= \gamma_j. \end{cases}$$

It is very easy to see that (γ'_j) and (ϕ'_j) are two families satisfying the relations (5.1) and (5.2). Therefore, applying the first part of rel. (5.5), we get:

$$\forall k < i, \ \ (\phi_{k+1}\gamma_k)(\phi_i\gamma_i) = (\phi'_k\gamma'_k)(\phi'_i\gamma'_i) \leq (\phi'_i\gamma'_i),$$

i.e.

$$\forall k < i, \ \ (\phi_{k+1}\gamma_k)(\phi_i\gamma_i) \leq \phi_i\gamma_i. \tag{5.6}$$

Then, by applying iteratively relation (5.6) to our problem, we get

$$\begin{aligned} R_iR_i &\leq \phi_i\gamma_i[(\phi_i\gamma_{i-1})(\phi_{i-1}\gamma_{i-2})\dots(\phi_3\gamma_2)]\phi_i\gamma_i(\phi_iM_{i-1}) \\ &\leq \dots \\ &\leq \phi_i\gamma_i\phi_i\gamma_i(\phi_iM_{i-1}) \\ &= n_in_i(\phi_iM_{i-1}) \\ &\leq n_i(\phi_iM_{i-1}) \\ &= \phi_i(\gamma_i\phi_iM_{i-1}) \end{aligned}$$

and finally,

$$R_iR_i \leq R_i.$$

Hence, R_i is idempotent. Similarly, we can also prove that $S_iS_i = S_i$. Therefore, M_i, N_i, R_i and S_i are filters. □

5.3 Properties

We have already said that there is no need for duality between the γ_i's and the ψ_i's. Anyway, an ASF *cannot be self dual,* since $N_i \neq M_i$. In the present section, some properties of the ASF's with respect to the composition product and the order relationship between operators are presented. We also deal with new filters, which are derived from the previous ones and which are called *transposed ASF's.* Lastly, *symmetrical alternating filters* are introduced, and some of their properties reviewed.

Proposition 5.7 (Absorption laws) *For $i, j \in \mathbb{Z}$ such that $1 \leq i \leq j$, we have the following relations:*

$$M_jM_i = M_j \leq M_iM_j \qquad R_iR_j \leq R_j = R_jR_i, \tag{5.7}$$

$$N_iN_j \leq N_j = N_jN_i \qquad S_jS_i = S_j \leq S_iS_j. \tag{5.8}$$

proof: We shall only proof (5.7), since the two relations (5.8) derive by duality. The following egality holds: $M_jM_i = m_jm_{j-1}\dots m_{i+1}M_iM_i$. Therefore, M_i being idempotent, $M_jM_i = m_jm_{j-1}\dots m_{i+1}M_i$, i.e. $M_jM_i = M_j$.

Let us show now the second part of this equation, i.e. $M_j \leq M_iM_j$: we can write $M_j = \gamma_jM_j = (\gamma_i\gamma_{i-1}\dots\gamma_1)\gamma_jM_j$. Now, $I \leq \phi_1$ (extensivity), which implies $\gamma_1 \leq \gamma_1\phi_1$. Thus, $\gamma_1 \leq \phi_2\gamma_1\phi_1$, which in turn implies the inequality $\gamma_2\gamma_1 \leq \gamma_2\phi_2\gamma_1\phi_1$. By iterating theis process, we finally obtain

$$\gamma_i\gamma_{i-1}\dots\gamma_1 \leq \gamma_i\phi_i\gamma_{i-1}\phi_{i-1}\dots\gamma_1\phi_1 = M_i.$$

Therefore, $M_j \leq M_i\gamma_jM_j$, i.e.

$$M_j \leq M_iM_j.$$

This completes the proof of the first relation.

Consider now the case of the R_i's and prove first that $R_jR_i = R_j$. Starting from $I \leq \phi_i$, we get $M_i \leq \phi_iM_i$ and, M_j being extensive, $M_jM_i \leq M_j\phi_iM_i$. But we just showed that $M_jM_i = M_j$, hence $M_j \leq M_jR_i$, which gives us finally

$$R_j = \phi_jM_j \leq R_jR_i.$$

Conversely, the operator R_jR_i can be decomposed in the following way:

$$R_jR_i = (\phi_j\gamma_j)[(\phi_j\gamma_{j-1})(\phi_{j-1}\gamma_{j-2})\dots(\phi_2\gamma_1)](\phi_i\gamma_i)\phi_iM_{i-1},$$

which is very close to the kind of decomposition used for proving the idempotence of the R_i's. In a similar way, we can iteratively apply rel. (5.6) to our equation. This yields finally

$$\begin{aligned} R_jR_i &\leq (\phi_j\gamma_j)[(\phi_j\gamma_{j-1})\dots(\phi_{i+1}\gamma_i)](\phi_i\gamma_i)\phi_iM_{i-1} \\ &\leq \phi_j[(\gamma_j\phi_j)\dots(\gamma_{i+1}\phi_{i+1})](\gamma_i\phi_i\gamma_i\phi_i)M_{i-1} \\ &= \phi_j[(\gamma_j\phi_j)\dots(\gamma_{i+1}\phi_{i+1})](\gamma_i\phi_i)M_{i-1} \\ &= \phi_jM_j = R_j. \end{aligned}$$

We therefore have $R_jR_i = R_j$.

The last inequality of rel. (5.7) is proved in a similar way as the previous one:

$$\begin{aligned} R_iR_j &= (\phi_i\gamma_i)[(\phi_i\gamma_{i-1})(\phi_{i-1}\gamma_{i-2})\dots(\phi_2\gamma_1)](\phi_j\gamma_j)\phi_jM_{j-1} \\ &\leq (\phi_i\gamma_i\phi_j\gamma_j)\phi_jM_{j-1} \\ &= (n_in_j)\phi_jM_{j-1} \\ &\leq n_j\phi_jM_{j-1} \\ &= \phi_jM_j = R_j \end{aligned}$$

This time again, we iteratively applied rel. (5.6), but we changed the i in this relation for j. There is a similar proof for the second part of rel. (5.8). $\square$

The transposed filters M_i^t, N_i^t, R_i^t and S_i^t are then defined as follows:

Definition 5.8 *For all $i \in \mathbb{Z}, i \geq 1$, the following operators:*

$$\boxed{\begin{array}{lllllll} M_i^t & = & m_1 m_2 \ldots m_{i-1} m_i & \quad & R_i^t & = & r_1 r_2 \ldots r_{i-1} r_i \\ N_i^t & = & n_1 n_2 \ldots n_{i-1} n_i & \quad & S_i^t & = & s_1 s_2 \ldots s_{i-1} s_i \end{array}}$$

are called alternating transposed filters of order i.

One can easily prove that these operators are filters: first, they are obviously increasing. Moreover, similar proofs as those given above for showing the idempotence of the ASF's may also be used here.

These new filters satisfy similar properties as the ASF's, namely:

Proposition 5.9 (Absorption laws) *For $i, j \in \mathbb{Z}$ such that $1 \leq i \leq j$, we have the following relations:*

$$M_j^t M_i^t \leq M_j^t = M_i^t M_j^t \qquad R_i^t R_j^t = R_j^t \leq R_j^t R_i^t, \tag{5.9}$$

$$N_i^t N_j^t = N_j^t \leq N_j^t N_i^t \qquad S_j^t S_i^t \leq S_j^t = S_i^t S_j^t. \tag{5.10}$$

The proofs are similar to that given above for the proposition 5.7.

All the above filters satisfy a great number of other properties that would be too long to detail here. However, it is interesting to introduce a last class of alterned filters that we call *alternating symmetrical filters,* and that are built from the two above class of filters:

Definition 5.10 *For all $i \in \mathbb{Z}, i \geq 1$, the following operators:*

$$\boxed{\begin{array}{lllllll} \tilde{M}_i & = & M_i^t M_i & \quad & \tilde{R}_i & = & R_i^t R_i \\ \tilde{N}_i & = & N_i^t N_i & \quad & \tilde{S}_i & = & S_i^t S_i \end{array}}$$

are called alternating symmetrical filters of order i.

Remark first that composing ASF's with their transposed versions in the opposite order would not have allowed us to derive new filters. Indeed, $M_i M_i^t = m_i$ and $N_i N_i^t = n_i$ (see prop. 5.3 above). Besides, the two relations (5.9) and (5.10) allow us to derive immediately the idempotence of the $\tilde{M}_i$'s and of the $\tilde{N}_i$'s, and thus to prove that we are dealing with filters (similarly, the $\tilde{R}_i$'s and $\tilde{S}_i$'s are also filters). One can say that these symmetrical filters "summarize" all the above properties of the alternating sequential filters and of the alternating transposed filters, since most of the inequalities that were proved before become equalities for them. More precisely:

Proposition 5.11 *For any $i, j \in \mathbb{Z}$, $1 \leq i$, $1 \leq j$, we have:*

$$\tilde{M}_i \tilde{M}_j = \tilde{M}_j \tilde{M}_i = \tilde{M}_{\sup(i,j)} \qquad \tilde{R}_i \tilde{R}_j = \tilde{R}_j \tilde{R}_i = \tilde{R}_{\sup(i,j)}, \tag{5.11}$$

$$\tilde{N}_i \tilde{N}_j = \tilde{N}_j \tilde{N}_i = \tilde{N}_{\sup(i,j)} \qquad \tilde{S}_i \tilde{S}_j = \tilde{S}_j \tilde{S}_i = \tilde{S}_{\sup(i,j)}. \tag{5.12}$$

proof: Suppose e.g. that $i \leq j$. Then

$$\begin{aligned}
\tilde{M}_i \tilde{M}_j &= M_i^t M_i M_j^t M_j \\
&= M_i^t (m_i m_{i-1} \ldots m_1 m_1 \ldots m_i m_{i+1} m_j) M_j \\
&= M_I^t (m_i m_{i+1} \ldots m_j) M_j \qquad \text{(rel. 5.3)} \\
&= M_j^t M_j .
\end{aligned}$$

Similarly, the equality $\tilde{M}_j \tilde{M}_i = M_j^t M_j$ holds. This completes the proof of the first relation of prop. 5.11. Similar proofs for the other equalities. □

The above proposition is at the basis of the following theorem, whose proof is straightforward:

Theorem 5.12 *The set $\{\tilde{M}_i, i \in \mathbb{Z}, i \geq 1\}$ (resp. $\{\tilde{N}_i, i \in \mathbb{Z}, i \geq 1\}$, $\{\tilde{R}_i, i \in \mathbb{Z}, i \geq 1\}$, $\{\tilde{S}_i, i \in \mathbb{Z}, i \geq 1\}$) constitutes a commutative semi-group of filters satisfying the following internal law of composition:*

$$\forall i \geq 1, j \geq 1, \qquad \tilde{M}_i \tilde{M}_j = \tilde{M}_{\sup(i,j)}.$$

Inside each of the four types of operations that we have described in this section (types M, N, R and S), there exist interesting order relationships. For example, as concerns type M, one can state the following succession of inequalities:

Proposition 5.13 *For any $i \in \mathbb{Z}, i \geq 2$, the following inequalities hold:*

$$\gamma_i \leq m_i \gamma_{i-1} \leq M_i \leq \left\{ \begin{array}{c} m_i \\ \tilde{M}_i \end{array} \right\} \leq M_i^t \leq \phi_{i-1} m_i \leq \phi_i .$$

proof: The first inequality is obvious: $\gamma_i \leq \gamma_{i-1}$ implies $\gamma_i = \gamma_i \gamma_i \leq \gamma_i \gamma_{i-1}$. Now, we also have $I \leq \phi_i$, hence $\gamma_i \leq \gamma_i \phi_i = m_i$. From this last relation, we get $\gamma_i \gamma_{i-1} \leq m_i \gamma_{i-1}$ and finally $\gamma_i \leq m_i \gamma_{i-1}$. The last inequality (i.e. $\phi_{i-1} m_i \leq \phi_i$) has a similar proof.

As concerns the second inequality, we first remark that for any $i \geq 1$, $\gamma_i \leq m_i \gamma_{i-1} \leq m_i m_{i-1} \gamma_{i-2} \leq \cdots \leq m_i m_{i-1} \ldots m_2 \gamma_1$. Thus, as $\gamma_1 \leq \gamma_1 \phi_1 = m_1$, we find the relation $\gamma_i \leq M_i$. Now, for any $i \geq 2$, this relation can be written $\gamma_{i-1} \leq M_{i-1}$, which implies $m_i \gamma_{i-1} \leq m_i M_{i-1} = M_i$, which is exactly the relation we had to prove. Similarly, $M_i^t \leq \phi_{i-1} m_i$.

The relation $M_i \leq m_i$ is relatively obvious, since (from rel. (5.4)) $\forall k < i, m_i m_k \leq m_i$. This implies:

$$\begin{aligned}
M_i &= m_i m_{i-1} \ldots m_1 \\
&\leq m_i m_{i-2} \ldots m_1 \\
&\leq \cdots \\
&\leq m_i m_1 \leq m_i .
\end{aligned}$$

Similarly, $m_i \leq M_i^t$.

Lastly, $M_i = m_i M_i$ implies, after successive applications of the relation (5.4),

$$\begin{aligned}
M_i &\leq m_{i-1} m_i M_i \\
&\leq m_{i-2} m_{i-1} M_i \\
&\leq \cdots \\
&\leq m_1 m_2 \ldots m_i M_i = M_i^t M_i = \tilde{M}_i .
\end{aligned}$$

Similarly, we get $\tilde{M}_i \leq M_i^t$, which completes the proof of the above succession of inequalities. □

We can also look for inequalities between filters of different types, and state the following proposition:

Proposition 5.14 *For any $i \in \mathbb{Z}, i \geq 1$, we have*

$$s_i \leq \left\{ \begin{array}{c} m_i \\ n_i \end{array} \right\} \leq r_i$$

and

$$S_i \leq \left\{ \begin{array}{c} M_i \\ N_i \end{array} \right\} \leq R_i \qquad \tilde{S}_i \leq \left\{ \begin{array}{c} \tilde{M}_i \\ \tilde{N}_i \end{array} \right\} \leq \tilde{R}_i. \tag{5.13}$$

Besides, the same kind of inequalities could also be written for the alternating transposed filters. At this point, we are certainly far from having written all the non-trivial possible order relations between alternating filters. However, the equalities and inequalities that we have proved lead us to a better understanding of the ASF's and tell us how they should be composed to get new filters. Moreover, we are now able to simplify the expression of a given filter ψ until the most *compact* expression is reached. This allows us to optimize the computation time of ψ.

Compatibility under magnification

When the two primitive families (γ_i) and (ϕ_i) are compatible under magnification, i.e.

$$\forall X, \forall k > 0, \quad \left\{ \begin{array}{lcl} \phi_k(kX) & = & k\phi_1(X), \\ \gamma_k(kX) & = & k\gamma_1(X), \end{array} \right. \tag{5.14}$$

this property is transmitted to the corresponding ASF's. The physical interpretation of this is clear: it means that the ASF of order k works on the k times magnified image exactly as does the ASF of order 1 on the initial image.

5.4 Applications, computation time

These alternating filters are among the most useful filters in mathematical morphology. They actually provide efficient filtering in many image cleaning problems, and can be finely adjusted to each case. Indeed, we can operate on:

- the families (γ_i) and (ϕ_i) (most of the time, this comes down to choosing families of structuring elements),
- the type of filter (alternating sequential filter, alternating transposed filter, alternating symmetrical filter...),
- the "size" of the filter,
- etc...

The only problem with such filters (and with most filters used for image cleaning) is that of the computation time. Although the formulas for computing these filters can be "compacted", a certain number of elementary operations has yet to be performed in each case. Thus, the computation time of an ASF may well be very long on a non specialized equipment.

As an example, suppose that the computation times of ϕ_i and γ_i are equal to $i \times \Delta t$, for a fixed image size. Then, the computation time of M_i, N_i, etc... is proportional to i^2, as shown by table 5.1 (we suppose that the most efficient fomulas are used):

Filter	Computation time	Computation time for $i = 5$
M_i, N_i, M_i^t, N_i^t	$i(i+1)\Delta t$	$30 \times \Delta t$
R_i, S_i, R_i^t, S_i^t	$i(i+2)\Delta t$	$35 \times \Delta t$
$\tilde{M}_i, \tilde{N}_i$	$2i(i+1)\Delta t$	$60 \times \Delta t$
$\tilde{R}_i, \tilde{S}_i$	$(2i^2+2i+1)\Delta t$	$65 \times \Delta t$

Table 5.1: Computation time of some sequential filters, provided that the time required for computing γ_i or ϕ_i equals $i \times \Delta t$ (for a fixed image size).

In order to reduce the above computation times, it is sometimes useful to introduce the following type of alternating filters [19, page 213]:

$$\begin{aligned} M_i(2) &= m_i m_{i-2} \ldots m_3 m_1 \\ M_i'(2) &= \phi_i \gamma_{i-1} \phi_{i_2} \ldots \gamma_2 \phi_1, \end{aligned}$$

i being here an odd number. These filters obey to similar absorption laws as those which were presented above. Moreover, their filtering capabilities are, in most concrete problems, practically as good as the capabilities of the alternating filters described in this chapter. However, they can be computed two times faster than the "regular" alternating sequential filters. This is extremely interesting in practice.

Chapter 6

Activity lattice, toggle mappings

6.1 Toggling, optimization and self duality

In chapter 1, we have seen that, given a set E, the two sets $\mathcal{P}(E) = \{X, X \subseteq E\}$ and $\mathcal{F}(E, \overline{\mathbb{R}}) = \{f, f : E \longrightarrow \overline{\mathbb{R}}\}$ are complete and distributive lattices. The first one is even complemented. Later, at chapter 3, we have associated with any complete lattice $\mathcal{T}$ the set $\mathcal{T}'$ of the increasing mappings applying $\mathcal{T}$ on itself, and shown that $\mathcal{T}'$ is in turn a complete lattice. When starting from $\mathcal{P}(E)$ or $\mathcal{F}(E, \overline{\mathbb{R}})$, we shall denote this second lattice $\mathcal{P}'$ or $\mathcal{F}'$ respectively. More generally, the set of all the (increasing *or not*) mappings $\psi : \mathcal{T} \longrightarrow \mathcal{T}$ is still a complete lattice $\mathcal{T}''$ (same approach as for $\mathcal{T}'$ in chapter 3), denoted $\mathcal{P}''$ and $\mathcal{F}''$ in the two cases of sets and of functions.

The idea developed in this chapter consists in associating with a function f (or with a set X):

1. *a* series *of possible transforms* $\psi_i f$,
2. *a* toggling criterion, *i.e. a decision rule which determines at each point* $x \in E$ *the "best" value among the "candidates"* $(\psi_i f)(x)$.

Such an approach allows us to introduce optimality conditions. The "best" choice may be understood in the sense of noise reduction, of contrast enhancement, or of both. It may also be understood, for multispectral images, in the sense of edge matching between channels, etc... The toggling approach corresponds to the various minimization techniques involved in linear processings (e.g. minus squares). But there is no quadratic form here. The candidate mappings ψ_i play the role of the Lagrange coefficients. Moreover, for minimizing the criterion, the inf is used rather than derivatives.

Toggle mappings are not necessarily self dual for the complement (set case) or for the negation (function case). However, thay can *always* admit a self dual version. It is obtained by taking the self dual family (ψ_i, ψ_i^*) generated by any family (ψ_i) of primitives, and by symmetrizing the toggling criteria with respect to duality. We shall give an explicit example of such a procedure in § 6.3.1, where *morphological centers* are concerned. In the following, we take it for granted that all the algorithms presented in this chapter possess the property of having self dual versions, and we shall not repeat this point after each result.

6.2 Activity lattice and semi-lattice

6.2.1 Boolean case

Let $\mathcal{P}(E)$ be a boolean lattice, and $\mathcal{P}'$, $\mathcal{P}''$ be the two associated operator lattices. Equip $\mathcal{P}''$ with the following relationships:

$$\forall \psi_1, \psi_2 \in \mathcal{P}'', \quad \psi_1 \preceq \psi_2 \Longleftrightarrow \begin{cases} I \cap \psi_1 & \supseteq & I \cap \psi_2, \\ I \cup \psi_1 & \subseteq & I \cup \psi_2. \end{cases} \tag{6.1}$$

Note that the second inequality may be rewritten

$$\Theta \cap \phi_1 \subseteq \Theta \cap \phi_2,$$

with Θ standing for the complement operator, i.e. $\Theta : X \longmapsto X^C$.)

Obviously, relation $\preceq$ is an ordering relationship. Moreover, it makes $\mathcal{P}''$ a complete lattice. Indeed, any minorant m of a family (ψ_i) in $\mathcal{P}''$ satisfies the following system:

$$\begin{cases} I \cap m & \supseteq & I \cap \zeta, \\ I \cup m & \subseteq & I \cup \eta, \end{cases} \tag{6.2}$$

with

$$\text{where } \zeta = \cup_i \psi_i \qquad \text{and} \qquad \eta = \cap_i \psi_i. \tag{6.3}$$

Therefore, if there exists a mapping β verifying

$$\begin{cases} I \cap \beta & = & I \cap \zeta \\ I \cup \beta & = & I \cup \eta, \end{cases} \tag{6.4}$$

then β is larger, for order relation $\preceq$, than any minorant m. Note that we can modify the second equality of (6.4) by using again the complement operator Θ:

$$I \cup \beta = I \cup \eta \iff \Theta \cap \beta = \Theta \cap \eta.$$

Now, system (6.4) admits one and one only solution, since every element $\alpha \in \mathcal{P}''$ may be decomposed into:

$$\alpha = (I \cup \Theta) \cap \alpha = (I \cap \alpha) \cup (\Theta \cap \alpha). \tag{6.5}$$

By applying this remarkable identity to β, we find:

$$\begin{aligned} \beta &= (I \cap \zeta) \cup (\Theta \cap \eta) \\ &= [(I \cap \zeta) \cup \eta] \cap [(I \cap \zeta) \cup \Theta] \\ \Longrightarrow \quad \dot{\wedge} \Psi_i = \beta &= (I \cup \eta) \cap \zeta = (I \cap \zeta) \cup \eta, \end{aligned} \tag{6.6}$$

a quantity which satisfies system (6.4)[1]. Similarly, if there exists a *smaller majorant* δ to the family (ψ_i), it must satisfy the system:

$$\begin{cases} I \cap \delta & = & I \cap \eta, \\ I \cup \delta & = & I \cup \zeta \Longleftrightarrow \Theta \cap \delta = \Theta \cap \zeta. \end{cases} \tag{6.7}$$

[1]In this chapter, the notations $\dot{\wedge}$ and $\dot{\vee}$ denote respectively the inf and the sup associated with the order relation $\preceq$ of lattices $\mathcal{P}''$ or $\mathcal{F}''$. They are called the *activity* inf and sup.

By using again identity (6.5), we find this time:

$$\dot{\vee}\psi_i = \delta = (\Theta \cup \eta) \cap \zeta = (\Theta \cap \zeta) \cup \eta. \tag{6.8}$$

Therefore, we have a lattice, which turns out to be complete and distributive (easy proof). We go from the inf $\dot{\wedge}$ to the sup $\dot{\vee}$ by replacing the identity I by the complementation Θ. As a matter of fact, the two operators I and Θ are just the null element and the universal element of the lattice generated by order $\preceq$. Applied to set $A \subseteq E$, the first one does not modify any point of A or of A^C, whereas the second one modifies all the points of A and of A^C.

Given two mappings ψ_1 and ψ_2 in $\mathcal{P}''$, we shall say that ψ_2 is more **active** than ψ_1 when $\psi_2 \succeq \psi_1$. By acting on a set $A \subseteq E$, ψ_2 removes more points from A than ψ_1 does, and also ψ_2 adds more points to A than ψ_1 does (which is nothing but the geometrical meaning of system (6.1)). This gives the following:

Theorem 6.1 *Let E be an arbitrary set, and let $\mathcal{P}''$ be the set of the mappings from $\mathcal{P}(E)$ into itself. The two inclusions*

$$\forall \psi_1, \psi_2 \in \mathcal{P}'', \quad \begin{cases} I \cap \psi_1 & \supseteq & I \cap \psi_2, \\ I \cup \psi_1 & \subseteq & I \cup \psi 2, \end{cases}$$

generate an ordering relation $\psi_1 \preceq \psi_2$ on $\mathcal{P}''$, to which is associated a complete distributive lattice. It is called the activity lattice $\mathcal{A}$. *The null element is the identity mapping I and the universal element is the complementation mapping Θ. If $\eta = \cap\psi_i$ and $\zeta = \cup\psi_i$ denote respectively the intersection and the union of a family (ψ_i) of elements of $\mathcal{P}''$, then the* inf *β and the* sup *δ of family (ψ_i) with respect to the activity are called* **center** *and* **anti-center** *of the ψ_i's and are given by the expressions:*

$$\begin{aligned} \dot{\wedge}\psi_i &= \beta = (I \cup \eta) \cap \zeta = (I \cap \zeta) \cup \eta, \\ \dot{\vee}\psi_i &= \delta = (\Theta \cup \eta) \cap \zeta = (\Theta \cap \zeta) \cup \eta. \end{aligned}$$

6.2.2 Function case

The activity ordering (6.1) applies obviously to the lattice $\mathcal{F}''$ of the mappings acting on $\mathcal{F}(E, \overline{\mathbb{R}})$, by changing the symbols $\subseteq$, $\supseteq$, $\cup$ and $\cap$ into $\leq$, $\geq$, $\vee$ and $\wedge$ respectively. The existence and the unicity of the activity inf β derive from the fact that $\mathcal{F}''$ is completely distributive [19, page 164]. Thus, $\mathcal{F}''$ turns out to be an inf semi-lattice for the activity order, but *no more*. As a set, the activity sup of subgraphs of functions exists, but it is not itself a subgraph (see Fig. 6.1).

The question which arises then is to know which conditions two mappings η and ζ in $\mathcal{F}''$, $\eta \leq \zeta$, must satisfy to define one and only one δ such that

$$\begin{cases} I \wedge \delta & = & I \wedge \eta, \\ I \vee \delta & = & I \vee \zeta. \end{cases} \tag{6.9}$$

(i.e. the equivalent of rel. (6.7)). For the sake of clarity, we shall introduce the following simplification in the notation:

> *when a condition applies to all the elements f of $\mathcal{F}(E, \overline{\mathbb{R}})$, function f is removed from the notation (For example, if at a given point $x \in E$, ηf must always be larger than or equal to f, we shall write $\eta_x \geq I_x$ instead of $\forall f \in \mathcal{F}(E, \overline{\mathbb{R}}), (\eta f)(x) \geq f(x)$.).*

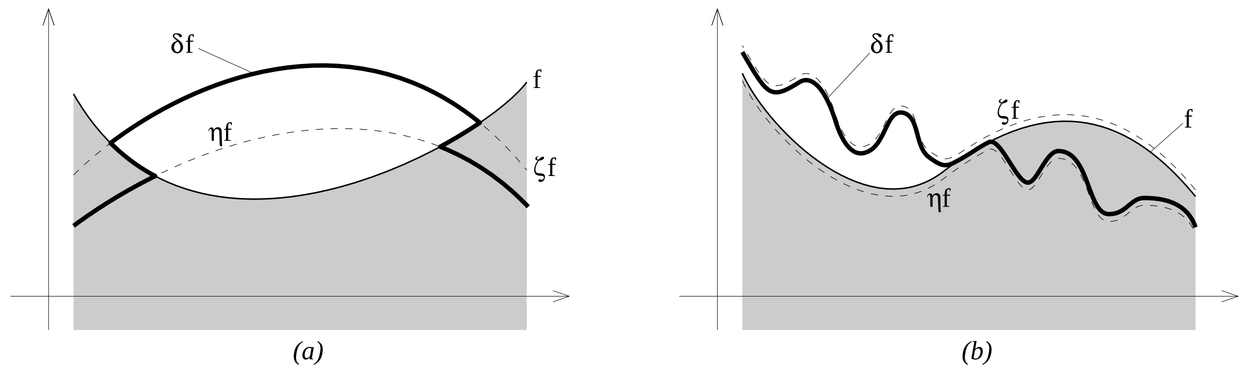


Figure 6.1: (a) The anti-center $\dot{\vee}$ of ζ and η may no longer be a function. (b) The conditions of theorem 6.2 are satisfied, so that, in this case, δf is a function.

Using this notation, we may state the following:

Theorem 6.2 *Let $\mathcal{F}(E,\overline{\mathrm{IR}})$ be the class of the functions $f : E \longrightarrow \overline{\mathrm{IR}}$, and $\mathcal{F}''$ be the family of the mappings from $\mathcal{F}(E,\overline{\mathrm{IR}})$ into itself. The elements of $\mathcal{F}''$ constitute an inf semi-lattice for the activity ordering $\preceq$. Moreover, the sup $\delta = \zeta\dot{\vee}\eta$ of an ordered pair (ζ,η) of $\mathcal{F}''$ exists if and only if:*

$$\{x \in E, \eta_x < I_x\} \cap \{x \in E, \zeta_x > I_x\} = \emptyset. \tag{6.10}$$

The activity sup δ is then given by

$$\forall x \in E, \quad \delta_x = \begin{cases} \eta_x & \textit{when} \quad \eta_x < I_x, \\ \zeta_x & \textit{when} \quad \zeta_x > I_x, \\ I_x & \textit{when} \quad \eta_x = \zeta_x = I_x. \end{cases} \tag{6.11}$$

proof: Easy, refer to [14]. □

In particular, we have:

Corollary 6.1 *Let τ and θ be respectively an extensive and an anti-extensive mapping from $\mathcal{F}(E,\overline{\mathrm{IR}})$ into itself. Their activity sup $\delta = \tau\dot{\vee}\theta$ maps also $\mathcal{F}(E,\overline{\mathrm{IR}})$ into itself if and only if, for every $f \in \mathcal{F}(E,\overline{\mathrm{IR}})$, the two domains*

$$X(f) = \{x \in E, (\theta f)(x) \neq f(x)\} \quad \textit{and} \quad Y(f) = \{x \in E, (\tau f)(x) \neq f(x)\}$$

are disjoined.

Finally, we draw two lessons from this section:

1. Any toggle mapping on $\mathcal{F}(E,\overline{\mathrm{IR}})$ using the activity ordering will also apply on $\mathcal{P}(E)$, since the former case is more restrictive. Therefore, in the following, we shall concentrate exclusively on $\mathcal{F}(E,\overline{\mathrm{IR}})$, $\mathcal{F}'$ and $\mathcal{F}''$.
2. The dissymmetry between the activity inf and sup suggests us to treat as a priority the simpler extremum, namely the inf, also called the center.

6.3 Morphological center

6.3.1 Geometrical interpretation

The center β of a family (ψ_i) in $\mathcal{F}''$ is itself a mapping on $\mathcal{F}(E,\overline{\mathbb{R}})$, i.e. an element of $\mathcal{F}''$, whose definition algorithm derives from relation (6.6):

$$\dot{\wedge}\psi_i = \beta = (I \vee \eta) \wedge \zeta = (I \wedge \zeta) \vee \eta, \tag{6.12}$$

with $\eta = \vee\psi_i$ and $\zeta = \wedge\psi_i$. Clearly, algorithm (6.12) is equivalent to the following system:

$$\begin{cases} I \vee \zeta & \leq & \beta & \leq & \zeta, \\ \zeta & \leq & \beta & \leq & I \vee \eta. \end{cases} \tag{6.13}$$

When applied, for example, to images $\psi_i(f)$ of a real-valued function f on E, this system states that if at point $x \in E$, all the $(\psi_i f)(x)$ are above $f(x)$, then we take the lowest value. If, on the other hand, they are all below $f(x)$, then we take the highest value. In all other cases, we leave $f(x)$ as it is (see Fig. 6.2). In particular, when $i = 2$, i.e. when ψ_1 and ψ_2 are the only primitives we have, then the morphological center is nothing but the median of I, ψ_1 and ψ_2. This bridges gap between Y. Neuvo approach [26] and the present one.

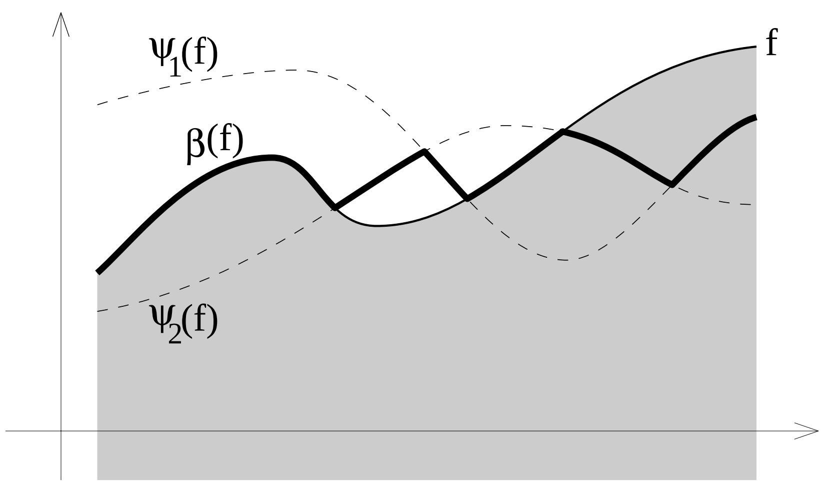


Figure 6.2: Morphological center β between two increasing mappings ψ_1 and ψ_2, when applied to the real-valued function f. We see the two properties that are common to all "central value mappings" (average, median, etc...) of an image: they wander between $\psi_1(f)$ and $\psi_2(f)$ and are closer to the initial function f.

We are able to produce as many self dual centers as we may desire. It is sufficient to start with a self dual family (ψ_i). In particular, to each element $\psi \in \mathcal{F}''$, there corresponds a self-dual element $\beta(\psi) \in \mathcal{F}'$ that is the inf $\psi\dot{\wedge}\psi^*$, where $\psi^*(f) = -\psi(-f)$. In fact,

$$\beta^* = [I \vee (\psi \vee \psi^*)]^* \wedge (\psi \wedge \psi^*)^* = [I \vee (\psi \wedge \psi^*)] \wedge (\psi \vee \psi^*) = \beta. \tag{6.14}$$

We will now carry on with increasing centers. Indeed, when the ψ_i's are increasing, we see from (6.12) that they transmit to β their growth property (as well as their possible planarity). Then, the center mapping has the advantage of not damaging high frequencies—as do convolutions—, that

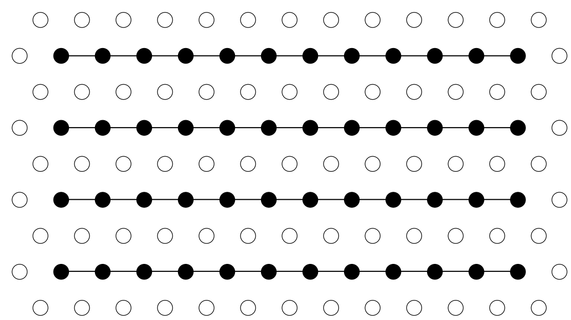


Figure 6.3: Oscillations of a median filter. The neighborhood B_p of pixel p consists in the seven points of the elementary hexagon $H(p)$ centered at point p. The transformation replaces $f(p)$ by the median of the histogram of values of $H(p)$. When acting on parallel lines, it oscillates...

of commuting with anamorphosis, etc... Now, the rank operators, and in particular the *median filtering* (see § 4.3), are also increasing and planar. However, they may oscillate under iteration (see Fig. 6.3) and even become periodic for some starting functions [19, page 160].

We shall try now to avoid this drawback by building up increasing centers which do not oscillate under iteration and tend toward morphological filters.

6.3.2 Iterations of increasing centers

A center β, of primitives (ψ_i), does not oscillate if and only if, for any function $f \in \mathcal{F}(E, \overline{\mathbb{R}})$, and for all points $x \in E$, we have

$$\begin{array}{ll} \text{either} & f(x) \leq (\beta f)(x) \leq (\beta \circ \beta f)(x) \leq (\beta^3 f)(x) \ldots \\ \text{or} & f(x) \geq (\beta f)(x) \geq (\beta \circ \beta f)(x) \geq (\beta^3 f)(x) \ldots \end{array}$$

In terms of operators, this point monotonicity means exactly that the activity of the successive powers β^n of β increase with n, i.e. $\forall n \geq 0, \beta^{n+1} \succeq \beta^n$. We now present two classes of β, where such an activity growth is satisfied. In practice, these two classes cover allmost all situations (for a general theory, see [19, page 166]).

6.3.2.1 ζ $\wedge$-overfilters, η $\vee$-underfilters

We always have

$$\beta \circ \beta = (\beta \wedge \zeta\beta) \vee \eta\beta = (I \wedge \zeta \wedge \zeta\beta) \vee (\eta \wedge \zeta\beta) \vee \eta\beta. \tag{6.15}$$

Moreover, when ζ is an $\wedge$-overfilter and η is a $\vee$-underfilter, the inequalities (6.13) imply that

$$\eta\beta \leq \eta = \eta(I \vee \eta) \leq \zeta = \zeta(I \wedge \zeta) \leq \zeta\beta,$$

and according to rel. (6.15), $\beta\beta = \beta$. The center β is thus a **morphological filter.** Such a case occurs, among others, when the ψ_i's are strong filters, since then, ζ is the sup of $\wedge$-filters and η the inf of $\vee$-filters. Moreover, when the ψ_i's are strong, β admits the following double decomposition:

$$\beta = (I \vee \eta)(I \wedge \zeta) = (I \wedge \zeta)(I \vee \eta) = \hat{\eta}\check{\zeta} = \check{\zeta}\hat{\eta}, \tag{6.16}$$

i.e., by theorem 4.4, it is itself a strong filter, called the *middle element* between η and ζ.

Examples:

- We have seen in § 4.4 that the composition product $\phi_{l,\alpha}\gamma_{l,\alpha}$ of the morphological opening $\gamma_{l,\alpha}$ and closing $\phi_{l,\alpha}$ with respect to a segment of length l and direction α is a strong filter (in $\mathbb{R}^n$ or in $\mathbb{Z}^n$). Put $\psi_\alpha = \phi_{l,\alpha}\gamma_{l,\alpha}$. Then, in $\mathbb{R}^2$ for instance, the family $(\psi_\alpha)_{\alpha\in[0,\pi]}$ admits a strong, isotropic and self-dual filter β as center. Note that, by starting from $\gamma_{l,\alpha}\phi_{l,\alpha}$, we obtain *another* center β', strong itself, isotropic and self-dual, but such that $\beta' \geq \beta$.

- We have also seen in § 4.3 that the median transformation m in a given neighborhood B was associated with the two envelopes

  $$\delta_B m \quad \text{and} \quad \varepsilon_B,$$

  where δ_B and ε_B stand for the Minkowski addition and subtraction respectively. Now, since $\delta_B m$ is a $\wedge$-overfilter and $\varepsilon_B m$ is a $\vee$-underfilter such that $\delta_B m \geq \varepsilon_B m$, then the center

  $$\beta = (I \wedge \delta_B m) \vee \varepsilon_B m$$

  is idempotent. Therefore, it is a morphological filter.

6.3.2.2 ζ $\vee$-underfilters, η $\wedge$-overfilters

Under these new assumptions, we have $\zeta \leq \zeta(I \vee \eta) \leq \zeta(I \vee \zeta) = \zeta$, and

$$\beta = (I \vee \eta) \wedge \zeta = (I \vee \eta) \wedge \zeta(I \vee \eta) = (I \wedge \zeta)(I \vee \eta),$$

as well as, by duality $\beta = (I \vee \eta)(I \wedge \zeta)$. This gives us

$$\beta^n = (I \wedge \zeta)^n (I \vee \eta)^n = (I \vee \eta)^n (I \wedge \zeta)^n.$$

We know, from theorem 2.6, that when the lattice $\mathcal{F}(E, \overline{\mathbb{R}})$ is finite, then for a certain $n_0 < +\infty$, $(I \vee \zeta)^{n_0} = \check{\zeta}$ and $(I \vee \eta)^{n_0} = \hat{\eta}$. Hence,

$$\beta^{n_0} = \check{\zeta}\hat{\eta} = \hat{\eta}\check{\zeta}.$$

The n_0-th iteration of β yields the middle element between ζ and η, in the sense of rel. (6.16). It is still a strong filter, since it is written as a *commutative* product of an opening and a closing (theorem 4.4). Moreover, we have for all n:

$$I \wedge \beta^n = (I \wedge \zeta)^n \quad \text{and} \quad I \vee \beta^n = (I \vee \eta)^n,$$

so that the successive powers of β are more and more active. When applied to a function f, the sucessive transforms $\beta^n f$ coincide with $(f \wedge \zeta f)^n$ in the zones where $(\beta f)^n$ decreases, and with $(f \vee \eta f)^n$ in the zones where $(\beta f)^n$ increases.

Example: It suffices to start from an arbitrary opening γ and an arbitrary closing ϕ, and to take $\zeta = \phi\gamma\phi$ and $\eta = \gamma\phi\gamma$. For small structuring elements (size one or two), the range of n_0 goes from two to five. For larger ones (e.g. size five), this range goes up to ten.

6.4 Toggle mappings

A toggle mapping ω is defined, on the one hand, by a family (ϕ_i) of reference mappings called *primitives,* and, on the other hand, by a *decision rule* which makes, at each point x, ω_x equal to one of the primitives $\psi_{i,x}$. The first example of a toggle mapping which comes in mind is that of the thresholding operation. The primitives are the white and the black, and the decision rule involves, at point x the value $f(x)$ and that of a constant, namely the threshold level. Note that thresholding is an idempotent operator. In this simple example, the two primitives are independent from the function f under study. But it is not always the case, and we have just seen toggle mappings such as the center, where the primitives are themselves transformations acting on the initial image. This leads us to the following formal definition:

Definition 6.3 *Let $\mathcal{F}(E,\overline{\mathbb{R}})$ be the class of the functions $f : E \longrightarrow \overline{\mathbb{R}}$, and $\mathcal{F}''$ be that of the mappings from $\mathcal{F}(E,\overline{\mathbb{R}})$ into itself. Given a family (ψ_i) of elements of $\mathcal{F}''$, one calls toggle mapping of primitives (ψ_i) any mapping ω of $\mathcal{F}''$ such that:*

(i) at each point x, ω_x equals one of the $\psi_{i,x}$ or I_x,

(ii) the criterion which affects one of the ψ_i's, say ψ_{i_0} to ω at a given point x depends only on the various primitives ψ_i, on the numerical value I_x and on possible constants,

(iii) if at point x, at least one of the ψ_i's, say ψ_{i_0}, coincides with the identity mapping I, then:

$$\omega_x = I_x = \psi_{i_0,x}. \tag{6.17}$$

Toggle mappings generate jumps, and the first way for keeping down this effect is to look for idempotent toggles. The following theorem, which involves the class $\mathcal{C}$ of the continuous functions $f : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$, provides a comprehensive class of such operators [21]:

Theorem 6.4 *Let, in $\mathbb{R}^n$, $\omega : \mathcal{C} \longrightarrow \mathcal{C}$ be a toggle mapping which admits as primitives the family $(\psi_j)_{j\in J}$ of strong filters, and let ρ and σ be the two mappings defined by:*

$$\begin{aligned} \rho_x &= \vee\{\psi_{j,x}, j \in J, \psi_{j,x} < I_x\}, \\ \sigma_x &= \wedge\{\psi_{j,x}, j \in J, \psi_{j,x} > I_x\}. \end{aligned}$$

If, for all points x, $\omega_x \in \{\rho_x, I_x, \sigma_x\}$ and if the family $(\psi_j)_{j\in J}$ is ordered for the order $\geq$, then the toggle mapping ω is idempotent.

Comments

- When dealing with the functions $f : \mathbb{Z}^n \longrightarrow \overline{\mathbb{Z}}$, the distinction between $\mathcal{C}$ and $\mathcal{F}(\mathbb{Z}^n, \mathbb{Z})$ is cumbersome.
- The middle element of equation (6.16) turns out to be a particular toggle, admitting the strong filters ψ_i, plus the identity, as primitives, and the activity inf as a criterion.
- Associate with ω the two following mappings:

$$\theta_x = \begin{cases} \rho_x & \text{when } \omega_x = \rho_x, \\ I_x & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \tau_x = \begin{cases} \sigma_x & \text{when } \omega_x = \sigma_x, \\ I_x & \text{otherwise.} \end{cases}$$

  Then, the conditions of corollary 6.1 are fulfilled, and the toggle ω is nothing but the activity sup $\tau\dot{\vee}\theta$ (whereas the ψ_i themselves do not have any activity sup).

6.4.1 First example: contrast mapping

Take for primitives an opening γ and a closing ϕ, and the criterion according to which, when $\gamma_x \neq I_x$ and $I_x \neq \phi_x$, the value at point x must move down (to γ_x) or up (to ϕ_x). This generates a *two states contrast* κ which is idempotent (theorem 6.4). In this simple case, the idempotence may be proved directly, since

$$\kappa \geq \gamma\kappa \geq \gamma\gamma = \gamma.$$

Hence, $\gamma_x = \kappa_x \Longrightarrow (\gamma\kappa)_x = \kappa_x$. Now, rel. (6.17), when applied to κ itself, implies:

$$(\gamma\kappa)_x = \kappa_x \quad \Longrightarrow \quad (\gamma\kappa)_x = (\kappa\kappa)_x,$$

i.e. by combining these two implications as well as their dual versions,

$$(\kappa_x = \gamma_x \ \text{ or } \ \kappa_x = \phi_x) \quad \Longrightarrow \quad (\kappa)_x = (\kappa\kappa)_x,$$

so that κ is idempotent. Fig. 6.4 presents an example of such a contrast algorithm (the closing and the opening have been replaced in this example by a dilation and an erosion of f).

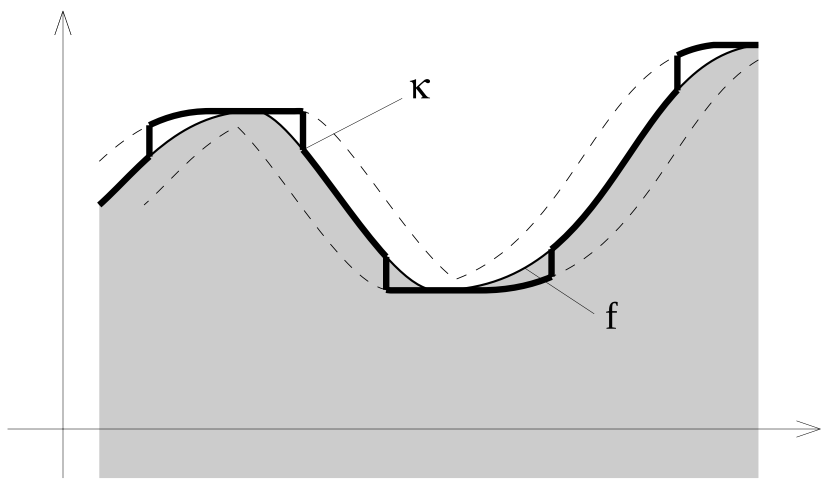


Figure 6.4: Example of contrast mapping.

6.4.2 Second example: combined toggle

In this second example, we will combine the antinomic properties of *noise reduction* of the centers and of *contrast enhancement.* Let ζ and η be two strong filters, and let (γ, ϕ) be a pair of opening and closing such that

$$\gamma \leq \eta \leq \zeta \leq \phi.$$

Fig. 6.5 illustrates the arrangement of the four transforms γf, ηf, ζf and ϕf around the initial function f. According to theorem 6.4, any toggle mapping ω is allowed to jump up (or down) at each point x, from $f(x)$ to the nearest transform above (or below) f. However, this flexibility is amended by one constraint: if at point x, f equals one of the four transforms under study, one cannot modify it.

As an efficient definition rule for ω, we may proceed as follows:

1. When $(\eta f)(x) \leq f(x) \leq (\zeta f)(x)$, apply a contrast κ, of primitives η and ζ.

2. When $(\gamma f)(x) \leq f(x) < (\eta f)(x)$, go down to $(\gamma f)(x)$, except if $(\zeta f - \eta f)(x)$ is smaller than a fixed value d. In the latter case, go up to $(\eta f)(x)$.

3. When $(\zeta f)(x) < f(x) \leq (\phi f)(x)$, apply a rule similar to rule 2, possibly with a scalar d' different from d.

The reason for this rule is the following. Usually, one takes $\zeta = \phi'\gamma'\phi'$ and $\eta = \gamma'\phi'\gamma'$, with an opening $\gamma' \geq \gamma$ and a closing $\phi' \leq \phi$. One can easily ascertain that ζ and η are close to one another in the narrow dark (resp. clear) features when both of them are also above (resp. below) the initial function. In the technique adopted here, such features are considered as non significant, and are reduced, if their sizes and their depths (their heights) are small enough with respect to the large opening γ and closing ϕ.

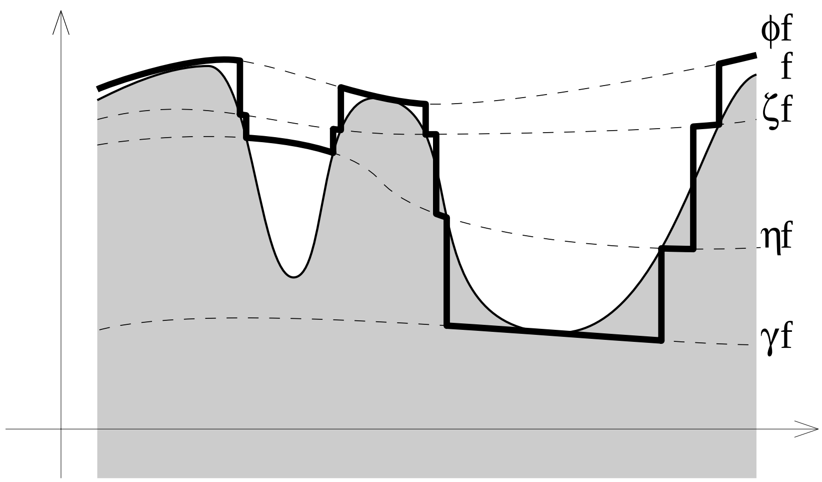


Figure 6.5: Function f, four primitives ϕf, ζf, ηf and γf. In superimposition, the toggle transform ωf given by the rules 1 to 3 detailed above.

6.5 Weakening of idempotence: the amplifier toggle mappings

We have wished the idempotence assumption because it ensures that the toggling process is well controlled. If not, we would run the two risks of generating, by iterations, oscillations and/or parasitic halos (especially around the peaks and slope changes, see [14]). However, some behaviours under iteration, such as the fixed zones growth, overcome the parasitic effects. The activity of a mapping ψ is said to exhibit a *fixed zone growth* when:

$$\psi \circ \psi \succeq \psi \quad \text{and} \quad \psi_x = I_x \Longrightarrow (\psi \circ \psi)_x = I_x. \tag{6.18}$$

In other words, the succesive iterations ψ_x, ψ_x^2,...,ψ_x^n may only strictly increase (or decrease) at point x, or stop. In the latter case, the stop becomes permanent through further iterations. Here is now an example of such a behaviour, which also shows a toggle mapping based on *one* primitive, and called *amplifier* (see Fig. 6.6).

Starting from a strong filter ψ, define the toggle mapping ω' by

$$\forall x \in E,\ \omega'_x = \begin{cases} \lambda'(I-\psi)_x + I_x & \text{when } I_x > \psi_x \quad (\Longrightarrow \omega'_x > I_x > \psi_x) \\ I_x & \text{when } I_x \leq \psi_x \quad (\Longrightarrow \omega'_x = I_x \leq \psi_x) \end{cases}$$

and similarly, the toggle mapping ω'' by

$$\forall x \in E,\ \ \omega''_x = \begin{cases} \lambda''(I-\psi)_x + I_x & \text{when } I_x < \psi_x, \\ I_x & \text{when } I_x \geq \psi_x. \end{cases}$$

Consider the composition product $\omega = \omega''\omega'$. When $\omega'_x < (\psi\omega')_x = \psi_x$, ω_x is equal to:

$$(\omega''\omega')_x = \lambda''(\omega' - \psi\omega')_x + \omega'_x = \lambda''(\omega' - \psi)_x + \omega'_x.$$

But $\omega'_x < \psi_x$ implies that $\omega'_x = I_x$, hence

$$(\omega''\omega')_x = \lambda''(I-\psi)_x + I_x = \omega''_x.$$

When $\omega'_x \geq (\psi\omega')_x$, then ω'' is the identity operator and

$$\omega_x = (\omega''\omega')_x = \omega'_x.$$

Finally, we obtain $\omega''\omega' = \omega''\dot{\vee}\omega'$, and by duality, we can state:

$$\omega = \omega''\omega' = \omega'\omega'' = \omega''\dot{\vee}\omega'. \tag{6.19}$$

Under iteration, the toggle ω exhibits a fixed zone growth behaviour, since

$$\omega\omega = \omega''\omega'\omega'\omega'' = \omega''[\lambda''(\lambda''+2)]\omega'[\lambda'(\lambda'+2)].$$

ω^2 is of the same type as ω, with changes in the intensity factors λ' and λ'' only. Therefore, the $\omega(\lambda', \lambda'')$'s satisfy a semi-group relationship when ψ is fixed, namely:

$$\omega(\lambda', \lambda'')\omega(\mu', \mu'') = \omega(\lambda' + \mu' + \lambda'\mu', \lambda'' + \mu'' + \lambda''\mu'').$$

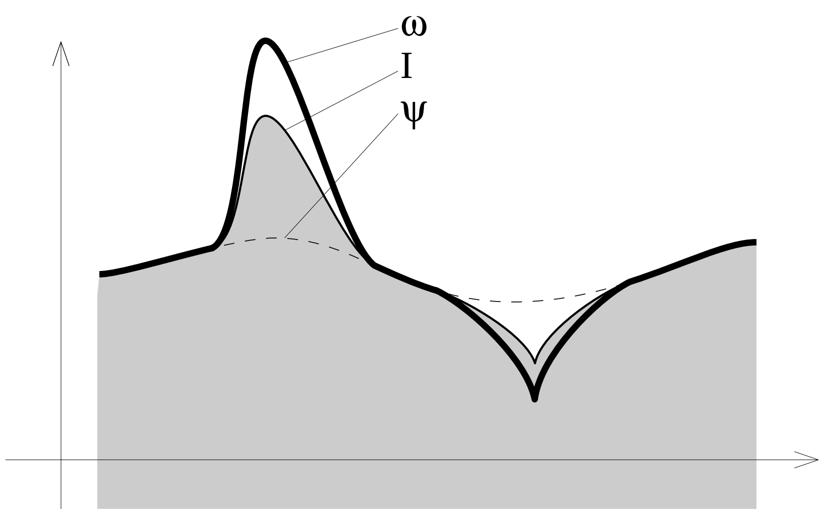


Figure 6.6: Example of an amplifier $\omega = \omega''\omega'$ based on a strong filter $\psi = \gamma\phi(I \wedge \gamma\phi)^n$ (with γ, ϕ morphological opening and closing).

Chapter 7

Segmentation of binary and grey-tone images

7.1 Introduction

Almost all the morphological transformations presented in the previous chapters were *increasing* ones. The operations that we shall introduce in this chapter will not necessarily share this characteristic. Moreover, the general framework within which we have been working until now is considerably particularized here: except § 7.2.1 and § 7.2.2, all the following sections are presented in the *discrete case.* We shall deal with *binary* images, i.e. subsets of $\mathcal{P}(\mathbb{Z}^2)$ and with *grey-tone* or *decimal* ones, i.e. functions of $\mathcal{F}(\mathbb{Z}^2, \mathbb{Z})$[1].

In this chapter, we are no longer concerned with morphological filters. After all, they constitute only a part of the field of mathematical morphology... Our present purpose is to show how morphology can be successfully applied to *segmentation* problems. Let us first say a couple of words on the meaning that we shall attach to the word "segmentation": in our sense, segmenting an image consists in *extracting* its different objects or regions and *contouring* them as precisely as possible. As concerns binary images, a very common segmentation problem is to separate its overlapping objects. Similarly, segmenting a grey-tone image comes most of the time down to dividing it into different regions (generally, one of these regions stands for the *background,* whereas the others correspond to the *objects*). In this chapter, starting from the concept of *marker,* we shall derive a general (and morphological) approach of segmentation problems. To reach this goal, a number of new transformations have to be presented first.

The first notions described here are that of *maximal balls* and *skeletons* (§ 7.2). They allow us to describe a set as a union of maximal balls and to define the *quench function,* whose interest for marking objects is discussed. The notion of *ultimate erosion* is then introduced. To go further, i.e. to really segment objects from their markers, we have then to describe *geodesic operations* (§ 7.3). Skeletons and geodesic transformations are jointly at the basis of a powerful binary segmentation algorithm. The last section is devoted to watersheds. They are defined for grey-tone images, and their interest for segmenting these images is particularly emphasized.

[1] However, when displaying examples of transformations on grey-tone images, we will use $\mathcal{F}(\mathbb{Z}, \mathbb{Z})$ rather than $\mathcal{F}(\mathbb{Z}^2, \mathbb{Z})$.

7.2 Skeletons

7.2.1 Skeletons and lattices

In this section, we consider a complete boolean lattice $\mathcal{T} = \mathcal{P}(E)$. To elaborate a concept of skeleton in such a framework (see [19, page 49]), we have to start from a family of structuring functions $(\delta_\lambda)_{\lambda\in[0,\lambda_0]}$ satisfying the following axioms:

(i) The class of the $(\delta_\lambda)_{\lambda\in[0,\lambda_0],x\in E}$ is closed for $\uparrow$ (monotonous continuity), and $\forall x \in E, \delta_0(x) = \{x\}$.

(ii) $\forall x \in E, \forall\lambda \geq 0, \mu \geq 0$, the following equality holds:

$$\delta_\lambda(x) = \varepsilon_\lambda \delta_{\lambda+\mu}(x), \tag{7.1}$$

where ε_λ is the dual erosion of δ_λ (recall from chapter 1 that we do not distinguish a dilation from its associated structuring function). The first axiom (which makes use of the notion of monotonous continuity, see [19, page 25]) states that for any family $(x_i, \lambda_i)_{i\in\mathbb{Z}^+}$ such that $\forall i, \lambda_i < \lambda_0$ and $i < j \Longrightarrow \delta_{\lambda_i}(x_i) < \delta_{\lambda_j}(x_j)$, there exists a $(x, \lambda) \in E \times [0, \lambda_0[$ such that

$$\delta_\lambda(x) = \vee_i \delta_{\lambda_i}(x_i).$$

As a consequence, consider a set $X \subseteq E$ such that for a given x_0, $X \subseteq \delta_{\lambda_0}(x_0)$. If the family (x_i, λ_i) satisfies

$$\forall i, \;\; \delta_{\lambda_i}(x_i) \subseteq X,$$

then $\vee_i \delta_{\lambda_i}(x_i)$ is still one $\delta_\lambda(x)$ which is also included in X. One can then show that each of the $\delta_{\lambda_i}(x_i)$ is contained in a *maximal* $\delta_\lambda(x)$. When E is the space $\mathbb{R}^2$ and when we deal with families of *balls*, $\delta_\lambda(x)$ is said to be a *maximal ball* of X (see Fig. 7.1).

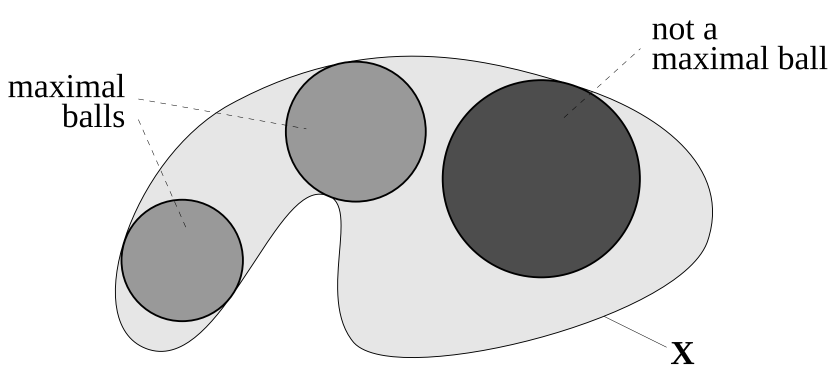


Figure 7.1: Examples of maximal balls.

In this framework, the *skeleton* of a given set $X \subseteq E$ is defined as follows:

Definition 7.1 *The skeleton of X is the set of the points x in E such that there exists a maximall ball $\delta_\lambda(x)$ included in X.*

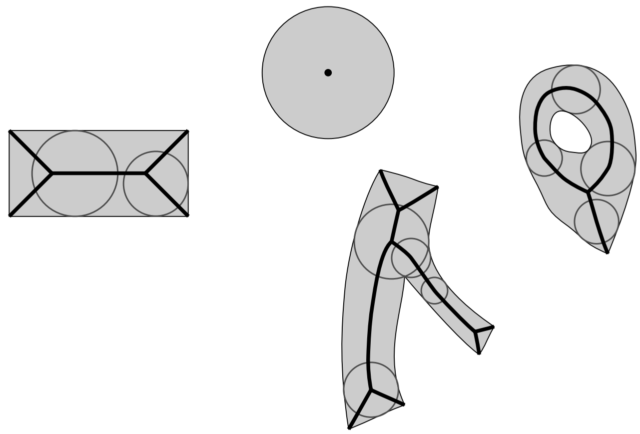


Figure 7.2: Some examples of skeletons.

Some examples of Euclidean skeletons (i.e. skeletons of sets $X \subset \mathbb{R}^2$ with respect to a family of *balls*) are presented on Fig. 7.2.

In order to give a compact formulation of the skeleton, let us now characterize the maximal "balls" of a given set X.

Proposition 7.2 *A $\delta_\lambda(x)$ is maximal in X if and only if the two following conditions are fulfilled:*

(i) $\delta_\lambda(x) \subseteq X$,

(ii) $\forall \alpha > 0, \forall y \in E, \delta_\lambda(x) \not\subseteq \delta_{\lambda+\alpha}(y) \subseteq X$.

Remark that the first condition is equivalent to $x \in \varepsilon_\lambda(X)$ (see chapter 1). Moreover, condition *(ii)* is equivalent to

$$\forall \alpha > 0, \forall y \in E \text{ such that } \delta_\alpha(y) \subseteq \varepsilon_\lambda(X), \quad x \notin \delta_\alpha(y) = \varepsilon_\lambda \delta_{\lambda+\alpha}(y).$$

Now, if $S_\lambda(X)$ denotes the set

$$S_\lambda(X) = \{x \in X, \delta_\lambda(X) \text{ maximal}\},$$

one can prove that

$$S_\lambda = \bigcap_{\alpha>0} \varepsilon_\lambda(X)/\gamma_\alpha \varepsilon_\lambda(X).$$

To summarize the results of this section, we can state the following theorem:

Theorem 7.3 *Let $(\delta_\lambda)_{\lambda \in \mathbb{R}^+}$ be a family of structuring functions on E such that the class $\{\delta_\lambda(x)\}_{\lambda<\lambda_0, x\in E}$ is inductive for inclusion (i.e. closed under $\uparrow$) and verifies:*

$$\forall x \in E, \forall \lambda, \mu \in]0, \lambda_0[, \ \ \varepsilon_\mu \delta_{\lambda+\mu}(x) = \delta_\lambda(x).$$

Then, the locus of the points $x \in E$ such that for a λ, $\delta_\lambda(x)$ is maximal in X defines the skeleton $S(X)$ of X, which is given by:

$$S = \bigcup_{0\leq\lambda\leq\lambda_0} S_\lambda = \bigcup_{0\leq\lambda\leq\lambda_0} [\bigcap_{\alpha>0} (\varepsilon_\lambda(X)/\gamma_\alpha \varepsilon_\lambda(X))]. \qquad (7.2)$$

Now, to every point $x \in S(X)$, we can associate the λ_x of the corresponding maximal element $\delta_{\lambda_x}(x)$. The mapping thus defined, i.e.

$$q_X \left(\begin{array}{lcl} S(X) & \longrightarrow & \mathbb{R}^+ \\ x & \longmapsto & \lambda_x \end{array} \right. \tag{7.3}$$

is called *quench function* (see Fig. 7.7). We will soon go back to the interest of this function for binary image segmentation. However, it should already be noticed that the data of a quench function q_X (with its support $S(X)$) is sufficient for synthesizing the set from where it was computed. Indeed, the following equality are proved easily:

$$X = \bigcup_{x \in S(X)} \delta_{q_X(x)}(x).$$

7.2.2 Some properties of the skeleton in the plane

In this section, we give a couple of results concerning the Euclidean skeleton, i.e. we consider the lattice $\mathcal{P}(\mathbb{R}^2)$, and the families of structuring functions that we use are families of closed balls $B_\lambda(x)$:

$$\forall x \in \mathbb{R}^2, \forall \lambda \geq 0, \;\; B_\lambda(x) = \{y \in \mathbb{R}^2, d(x,y) \leq \lambda\}, \tag{7.4}$$

where d designates the Euclidean distance in the plane. The skeleton seems to be a very natural notion, that does not hurt common sense. One could think that this so-called *medial axis* would satisfy many strong properties. However, it turns out that so many special cases and counter-examples appear that it is very difficult to prove prove nice theorems for this object. The results and examples given in this section are due to G. Matheron and can be found in [19, chapter 11].

Let us first review some counter-examples that illustrate the difficulties which appear when dealing with skeletons, and particularly with skeletons of closed sets:

1. The skeleton $S(X)$ is not necessarily connected, even though the set X is connected (see Fig. 7.3).

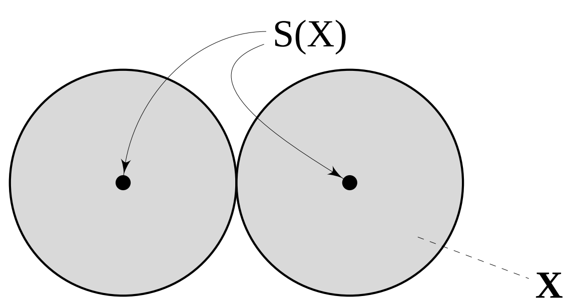


Figure 7.3: The skeleton of a connected set is not necessarily connected.

2. The skeleton $S(X)$ of a closed set X is not necessarily a closed set itself.
3. The skeleton transformation is neither upper- nor lower semicontinuous, as illustrated by Fig. 7.4.

Now, for open sets of $\mathbb{R}^2$, some interesting results can be proved, the most interesting being the two following ones:

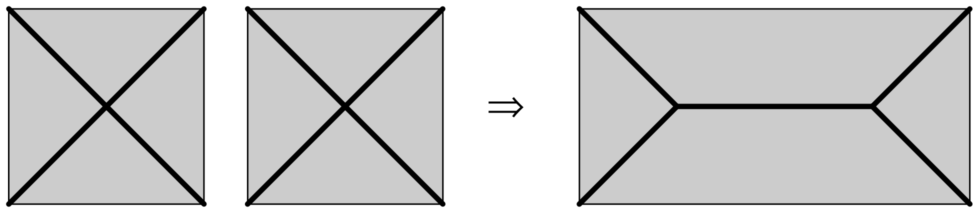


Figure 7.4: The skeleton transformation is neither u.s.c. nor l.s.c.

Theorem 7.4 *The mapping $X \longrightarrow \overline{S(X)}$ is a lower semi-continuous mapping from the set $\mathcal{O}$ of the open sets of $\mathbb{R}^2$ into that of its closed sets.*

Remark that this theorem only works for the *adherence* $\overline{S(X)}$ of X...

Theorem 7.5 *Let X be a connected open set of $\mathbb{R}^2$ containing no half-spaces. Then, the adherence $\overline{S(X)}$ of its skeleton is also connected.*

Many other more or less spectacular results concerning the skeleton can be found, for example, in [19, chapters 11–12].

7.2.3 Skeletons of binary images and derived notions

From now on, we stop dealing with general lattices or continuous spaces. Our purpose is to present notions in a very practical and natural way, and to orient the reader into a general approach of image segmentation problems via morphology [3]. When presenting "digital" formulas, we use an hexagonal grid (see Fig. 7.5) and denote H the elementary hexagon whereas the hexagon of size n is simply denoted nH. $H(p)$ and $nH(p)$ designate hexagons centered at pixel p, and of size 1 and n respectively. For the sake of clarity, the figures that we present refer to the continuous plane $\mathbb{R}^2$, whereas the corresponding notion is defined digitally.

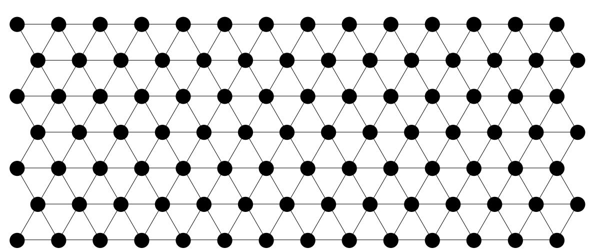


Figure 7.5: A portion of an hexagonal grid.

The notion of maximal ball extends to the case of a digital grid, and e.g. in the hexagonal case, an hexagon $nH(x)$ is said to be *maximal* in X if and only if there exists no other hexagon $H' \subseteq X$ such that $nH(x) \subseteq H'$ and $nH(x) \neq H'$. The skeleton of X is then the locus of the centers of the maximal hexagons included in X. The formula (7.2) given above can be applied to the digital case, and we may now state:

$$S(X) = \bigcup_{n=0}^{+\infty} [(X \ominus nH)/\gamma_H(X \ominus nH)]. \tag{7.5}$$

Remark that instead of hexagons, dodecahedrons or other digital approximations of discs can be used successfully. The same thing is also valid for square grids.

Equation (7.5) is at the basis of a very simple algorithm for determining skeletons in the digital case. One could think that being in a discrete space would leave behind all the "problems" discovered in § 7.2.2. Unfortunately, this is not the case, since the digital skeletons provided by equation 7.5 are said to be *unconnected,* i.e. they may well not be connected even though the initial set is. As these skeletons do not have the same *homotopy* as the initial set, they are not very useful in practice, and skeletons based on other types of algorithms are preferred: the most classical ones are based on successive *thinnings* (a morphological transformation that we have not presented here, and whose definition can be found in [18]) of sets until stability is reached [13]. Other algorithms, introduced by F. Meyer [19, chapter 13], are based on the computation of the distance function of the set under study and on the determination of the *crest lines* of this function (see 7.2.4).

As the transposition of the skeleton to digital grids is not so easy as it may seem, that of the *quench* function (sometimes also called *extinction* function) is straightforward. This function, whose support is the (unconnected) skeleton $S(X)$ of a set X associates with each pixel p of $S(X)$ the size of the corresponding maximal hexagon:

$$q_X \left(\begin{array}{lll} S(X) & \longrightarrow & \mathbb{Z}^+ \\ p & \longmapsto & n, nH(p) \text{ maximal in } X. \end{array} \right.$$

An example of quench function is presented in Fig. 7.7.

The quench function can be used for image compression, since the data of $S(X)$ and q_X is equivalent to that of X. Apart from that, the two main applications of the digital quench function are the following: first, it allows one to define very easily new image transformations. Consider for example a function $\psi : \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$. If rather than associating with each pixel p of $S(X)$ the radius $q_X(p)$ of the corresponding hexagon, we associate with it the radius $\psi(q_X(p))$, we define a new set transformation as follows:

$$\psi(X) = \bigcup_{p \in S(X)} \psi(q_X(p))H(p). \tag{7.6}$$

As an illustration of this, hexagonal erosions, dilations and openings of any size can be determined this way (see table 7.1).

Transformation	Associated ψ
$X \ominus nH$	$\psi(m) = -1$ if $m < n$ $= q_X(m) - n$ otherwise
$X \oplus nH$	$\psi(m) = q_X(m) + n$
$\gamma_{nH}(X)$	$\psi(m) = -1$ if $m < n$ $= q_X(m)$ otherwise

Table 7.1: How to determine the hexagonal erosion, dilation and opening of size n of a set X from $(S(X), q_X)$ (in this table, an hexagon of size -1 stands for $\emptyset$).

The other major application of the quench function is the definition of the *ultimate erosion* of a set, as presented in the next section.

Before dealing with this notion, let us define a last transformation, namely the *SKeleton by Influence Zones* or *SKIZ.* Given a set X made of n connected components $(X_i)_{1 \leq i \leq n}$, the influence zone $Z(X_i)$ of X_i is the locus of the points which are closer to it than to any other connected component of X:

$$Z(X_i) = \{p \in \mathbb{Z}^2, \forall j \neq j, d(p, X_i) \leq d(p, X_j)\}. \quad (7.7)$$

The distance that is used in this equation is a discrete one, defined on the hexagonal grid. Most of the time, we use the hexagonal distance d_H: $d_H(p_1, p_2) = n$ if and only if the length of the shortest paths between p_1 and p_2, whose edges are included in the grid, is equal to n (see Fig. 7.5).

The SKIZ of set X is the set of the boudaries of the influence zones $(Z(X_i))_{1 \leq i \leq n}$. An example of SKIZ is presented in Fig. 7.6. One can show that the skeleton by influence zones is a subset of the skeleton $S(X^C)$ of the *background* of X, i.e. of X^C. In practice, it is often determined by removing the irrelevant edges of $S(X^C)$, which are called *parasitic barbs.* The SKIZ will be very useful for the binary segmentation algorithm presented in § 7.3.5.

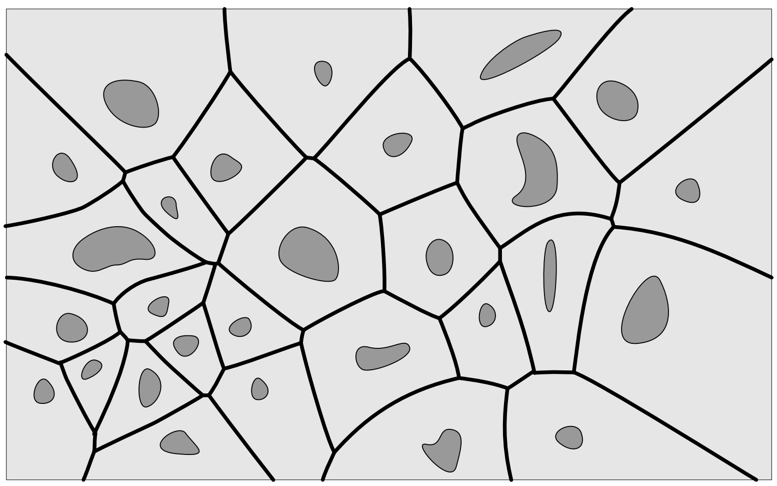


Figure 7.6: Example of skeleton by influence zones.

7.2.4 Ultimate erosion

The notions that we presented in the previous sections will now be applied to the resolution of the binary segmentation problem consisting in separating overlapping objects. Facing such a problem, the first thing to do is to find **markers** of the objects to extract. This is actually a required step in all the binary or decimal segmentation problems: marking the objects present in an image comes down to do exactly what a human being would do if he was told to point out with his finger all the objects of the image.

To reach this goal, the idea is to make use of the description of a set X in terms of maximal balls, and to look at the places where the largest maximal balls can be included in X. Consider for instance the simple case of a set X made of two overlapping discs (see Fig. 7.7). So as to point out the two different "components" of X, we look at the points $p \in S(X)$ corresponding to the "largest " maximal balls. Obviously, there are only two points corresponding to these largest balls: the **maxima** of the quench function of X, which are in this case located at the centers of the two discs composing X (besides, these two points correspond here to the extremities of $S(X)$, but this is far from being always the case). These maxima build the *ultimate erosion* of X.

Definition 7.6 *The ultimate erosion Ult(X) of a set X is the set of the maxima of the quench function q_X.*

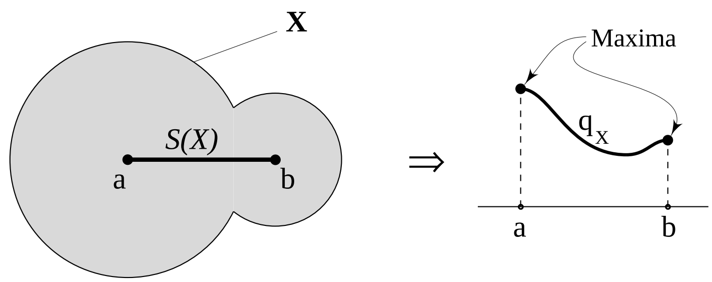


Figure 7.7: Skeleton of a set X, associated quench function q_X and its maxima.

There is obviously an inherent problem in this definition: how shall we define the maxima of a function whose support $S(X)$ is sometimes disconnected? To solve this problem, we propose now a different way for defining the ultimate erosion of a set X. This new definition is based on the following remark: when performing successive elementary erosions on a set, each of its "components" tends to become more *individual* before it disappears, as shown in Fig. 7.8. Suppose now that we are able to keep this components just before they disappear (we will see in § 7.3.3 how to do that). Their set will exactly build the ultimate erosion of X (see Fig. 7.9). This procedure is of course meaningless outside of the digital framework within which we chose to restrict ourselves.

The method that we just introduced suggests another way of determining the ultimate erosion of a set. Indeed, there is a morphological transformation which synthesizes all the information contained in the successive erosions of a set X. This transformation is called *distance function* and associates with each pixel p of X its distance to the backgroud:

$$\mathrm{dist}_X \left(\begin{array}{ccc} X & \longrightarrow & \mathbb{Z}^+ \\ p & \longmapsto & d(p, X^C) \end{array} \right) \tag{7.8}$$

An example of a distance function is shown in figure 7.10. According to our problem, the maxima of the distance function are exactly those of the quench function, i.e. they build the ultimate erosion of X. Moreover, computing $\mathrm{Ult}(X)$ from dist_X can be done very fast, since there exist extremely efficient algorithms for computing distance functions in digital images.

Moreover, as was said in § 7.2.3, the distance function is sucessfully used for determining skeletons. In this approach, the distance function is considered as a relief on which special pixel configurations, the *crest points* are selected. In a second step, these points are connected using an *upstream generation* algorithm. This method is very efficient and can be generalized to the case of *decimal skeletons.* The resulting skeletons are well drawn and have a one or two pixel thickness.

7.3 Geodesic transformations

At this point, we have solved the first step of our binary segmentation problem: we possess markers of our objects. The purpose of the present section is to introduce the morphological tools that are necessary for solving the second step, i.e. for coutouring our objects as precisely as possible. These tools are the *geodesic transformations* [8], which differ from the so-called Euclidean ones (i.e. the classical operations with hexagons, for instance) in that the underlying space is no longer $\mathbb{Z}^2$, but a given subset X of this space.

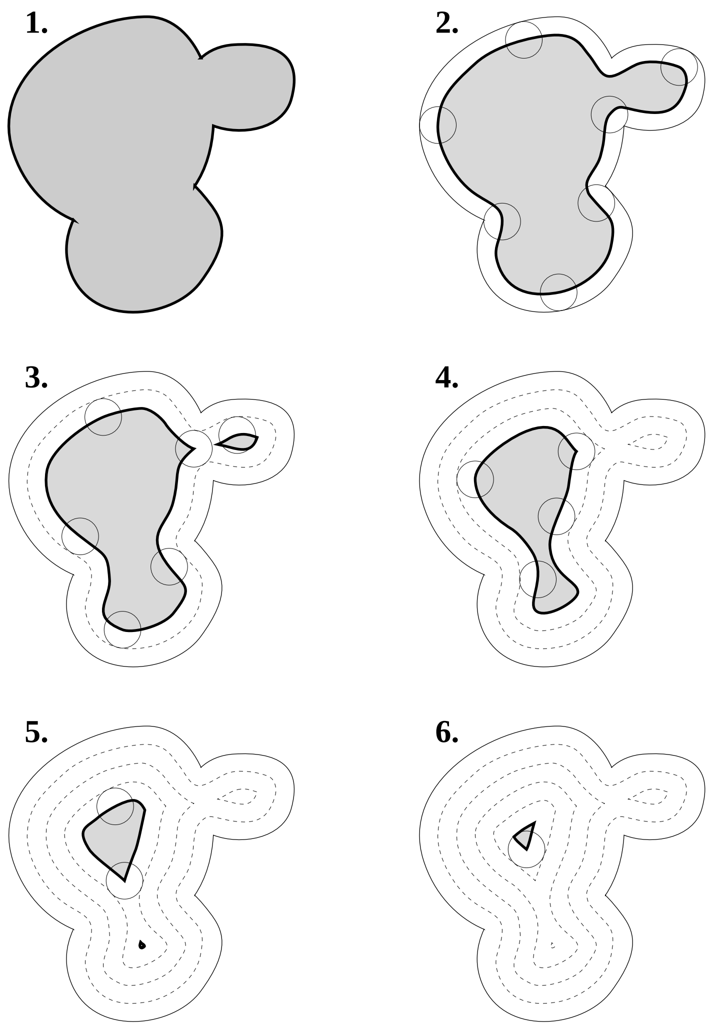


Figure 7.8: Successive erosions of a set. Each “component” tends to become more *individual* before it disappears.

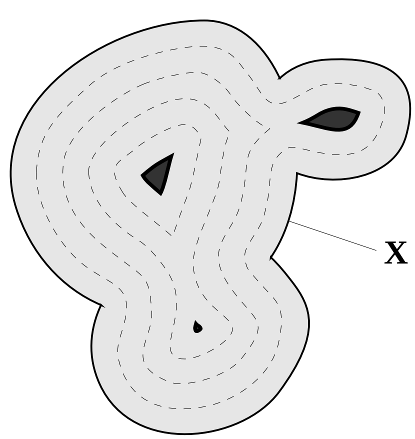


Figure 7.9: Ultimate erosion of a set.

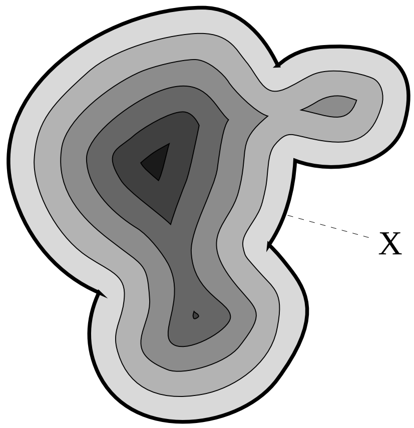


Figure 7.10: Example of distance functions. The darker it is, the larger the corresponding distance.

7.3.1 Geodesic distance

Let X be a set in $\mathbb{Z}^2$. We define the geodesic distance between two points p_1 and p_2 of X as the infimum of the length of the paths between p_1 and p_2 in X (if there are such paths at all):

$$d_X(p_1, p_2) = \inf\{l(C_{p_1,p_2}), C_{p_1,p_2} \text{ path between } p_1 \text{ and } p_2 \text{ included in } X\}. \tag{7.9}$$

Remark that this distance is a generalized one, in the sense that we put conventionally $d_X(p_1, p_2) = +\infty$ when p_1 and p_2 are in different connected components of X. The definition of d_X is illustrated in Fig. 7.11

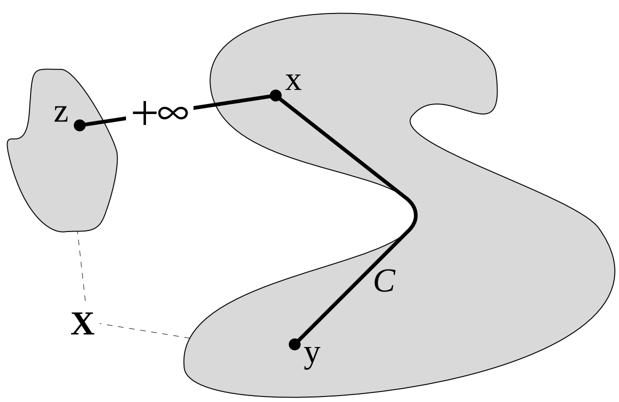


Figure 7.11: Geodesic distance in a set X.

We call *geodesic ball* of radius $n \in \mathbb{Z}^+$ and of center $p \in X$ the set $B_X(p, n)$ defined by:

$$B_X(p, n) = \{p' \in X, d_X(p', p) \leq n\}. \tag{7.10}$$

7.3.2 Geodesic dilations and erosions

Suppose now that X is equipped with its associated geodesic distance d_X. Given a $n \in \mathbb{Z}^+$, we consider the structuring function which associates with each pixel $p \in X$ the geodesic ball $B_X(p,n)$ of radius n centered in p. This allows us to define the *geodesic dilation* of a subset Y of X in the following way:

Definition 7.7 *The geodesic dilation $\delta_X^{(n)}(Y)$ of size n of set Y inside set X is given by*

$$\delta_X^{(n)}(Y) = \bigcup_{p \in Y} B_X(p,n) = \{p' \in Y, \exists p \in Y, d_X(p',p) \leq n\}. \tag{7.11}$$

The dual formulation of the geodesic erosion of size n of Y inside X is the following:

$$\varepsilon_X^{(n)}(Y) = \{p \in Y, B_X(p,n) \subseteq Y\} = \{p \in Y, \forall p' \in X/Y, d_X(p,p') > n\}. \tag{7.12}$$

Examples of geodesic dilation and erosion are shown in Fig. 7.12.

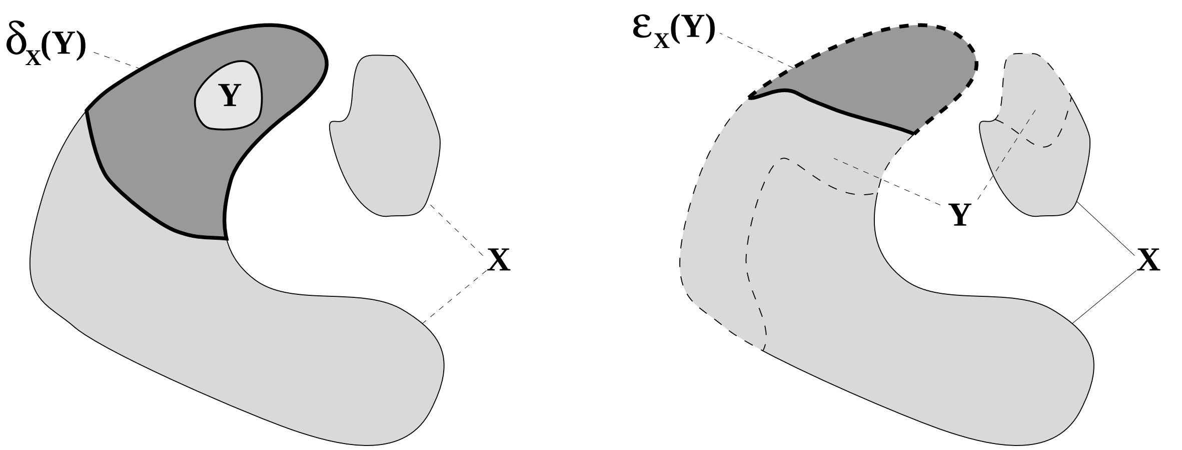


Figure 7.12: Examples of a geodesic dilation and of a geodesic erosion of set Y inside set X.

As already remarked above, the result of a geodesic operation on a set $Y \subseteq X$ is always included in X, which is our new workspace. As far as implementation is concerned, an elementary geodesic dilation (of size 1) of a set Y inside X is obtained, in the hexagonal case, by intersecting the result of a (Euclidean) dilation of size 1 of Y with the workspace X:

$$\delta_X^{(1)}(Y) = (Y \oplus H) \cap X. \tag{7.13}$$

A geodesic dilation of size n is obtained by iterating n elementary geodesic dilations:

$$\delta_X^{(n)}(Y) = \underbrace{\delta_X^{(1)}(\delta_X^{(1)}(\ldots \delta_X^{(1)}(}_{n \text{ times}}Y))). \tag{7.14}$$

(By duality, geodesic erosions are easily determined).

7.3.3 Reconstruction and applications

One can remark that by performing successive geodesic dilations of a set Y inside a set X, it is impossible to intersect a connected component of X which did not initially contain a connected component of Y. Moreover, in this successive geodesic dilations process, we progressively "reconstruct" the connected components of X that were initially *marked* by Y. This is shown in figure 7.13.

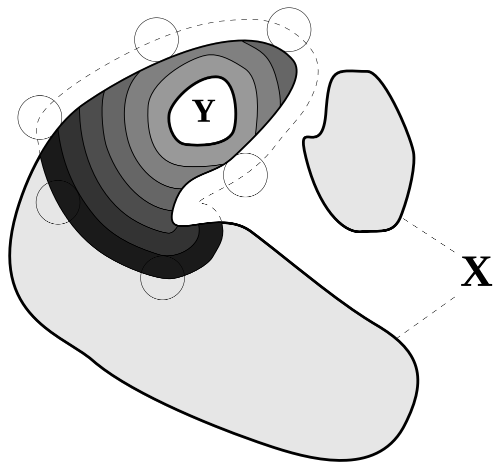


Figure 7.13: successive geodesic dilations of set Y inside set X.

Now, the sets with which we are concerned are finite ones. Therefore, there exists a n_0 such that

$$\forall n > n_0, \;\; \delta_X^{(n)}(Y) = \delta_X^{(n_0)}(Y).$$

At step n_0, we have entirely reconstructed all the connected components of X which were initially marked by Y. This operation is naturally called *reconstruction:*

Definition 7.8 *The reconstruction $r_X(Y)$ of the (finite) set X from set Y is given by the following formula:*

$$r_X(Y) = \lim_{n \to +\infty} \delta_X^{(n)}(Y). \tag{7.15}$$

Fig. 7.14 illustrates this transformation. Clearly, it will be extremely useful for reconstructing a set (or part of a set) from its markers. As an example, consider the first procedure suggested in § 7.2.4 for determining the ultimate erosion of a set X (this method is illustrated in Fig. 7.8). When presenting this algorithm, we assumed that, at each step n, we were able to keep the connected components of $X \ominus H$ that disappeared at step $n+1$, a requirement which can easily be expressed in terms of reconstruction operation: these connected components are exactly those which belong to the set $(X \ominus nH)/r_{X \ominus nH}(X \ominus (n+1)H)$. Finally, we obtain the formula:

$$\mathrm{Ult}(X) = \bigcup_{n \in \mathbb{Z}^+} [(X \ominus nH)/r_{X \ominus nH}(X \ominus (n+1)H)]. \tag{7.16}$$

Sometimes however, it is compulsory not to connect markers inside set X (this is the case for instance of the binary segmentation problem we are concerned with). In such cases, the *geodesic influence zones* of the connected components of set Y inside X are used. Indeed, the notions of influence zones and of SKIZ are very easily extended to the geodesic case, as shown by Fig. 7.15.

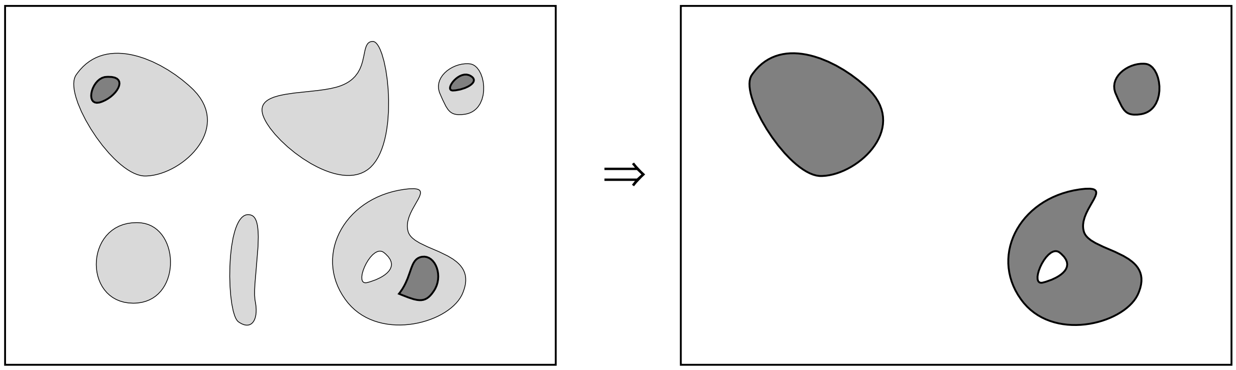


Figure 7.14: Reconstruction of X (light set) from Y (dark set).

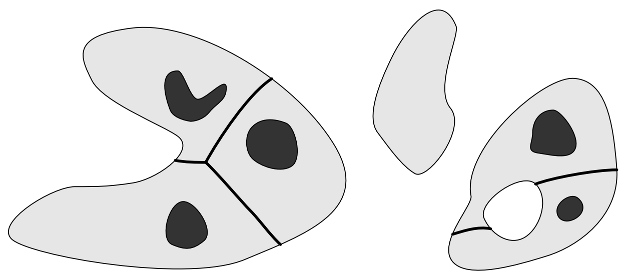


Figure 7.15: Example of geodesic SKIZ.

7.3.4 Geodesic operations for grey-tone images

At present, all the tools required for solving our binary segmentation problem are available. However, in the grey-tone case, a few more tools will be necessary, namely the geodesic decimal erosions and dilations. Therefore, it seems rather convenient to present these operations just after the corresponding binary ones.

The digital formulas (7.13) and (7.14) can easily be extended to the case of grey-tone functions defined on a digital grid. Given two functions f and g in $\mathcal{F}(\mathbb{Z}^2, \mathbb{Z})$, such that $f \leq g$, the geodesic dilations of size 1 and of size $n > 1$ of g with respect to f are defined by:

$$\delta_f^{(1)}(g) = (g \oplus H) \wedge f, \quad (7.17)$$

$$\delta_f^{(n)}(g) = \underbrace{\delta_f^{(1)}(\delta_f^{(1)}(\ldots\delta_f^{(1)}(g)))}_{n \text{ times}}. \quad (7.18)$$

By duality, when $g \geq f$, geodesic decimal erosions of g with respect to f are defined in the following way:

$$\varepsilon_f^{(1)}(g) = (g \ominus H) \vee f, \quad (7.19)$$

$$\varepsilon_f^{(n)}(g) = \underbrace{\delta_f^{(1)}(\delta_f^{(1)}(\ldots\varepsilon_f^{(1)}(g)))}_{n \text{ times}}. \quad (7.20)$$

These two transformations are presented in Fig. 7.16. They will turn out to be extremely useful when associated with *watersheds* (see § 7.4).

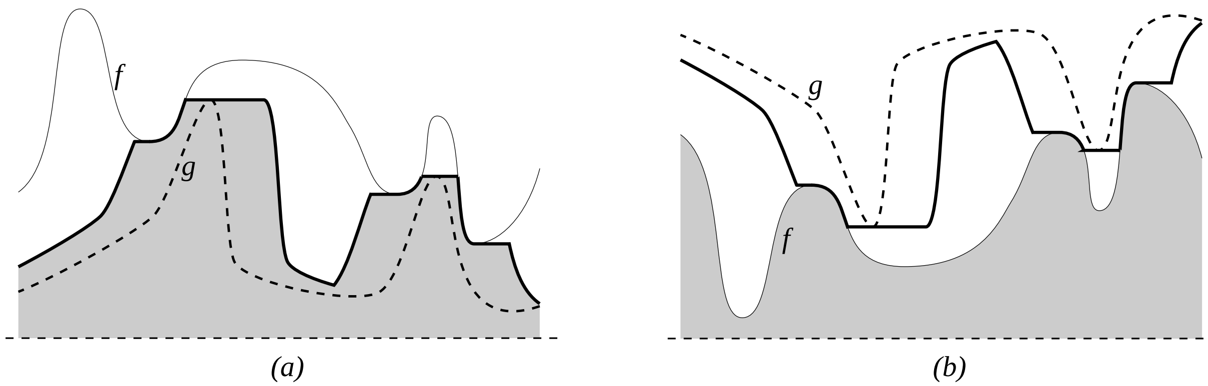


Figure 7.16: Examples of grey-tone geodesic dilation (a) and erosion (b).

Lastly, the concept of reconstruction is easily extended to the decimal case. Given two functions f and g in $\mathcal{F}(\mathbb{Z}^2, \mathbb{Z})$ such that $f \leq g$, the reconstruction of f from g is defined by:

$$r_f(g) = \lim_{n \to +\infty} \delta_f^{(n)}(g). \tag{7.21}$$

This transformation, which is shown in Fig. 7.17 is constantly useful in mathematical morphology. Among other things, it is frequently used in image filtering. Indeed, it is very easy to prove that the transformation ψ_n defined by

$$\psi_n(f) = r_f(f \ominus H)$$

is a morphological filter. Moreover, by subtracting $\psi_n(f)$ from the initial function f, we get a powerful tool for extracting the light and thin areas of this grey-tone image.

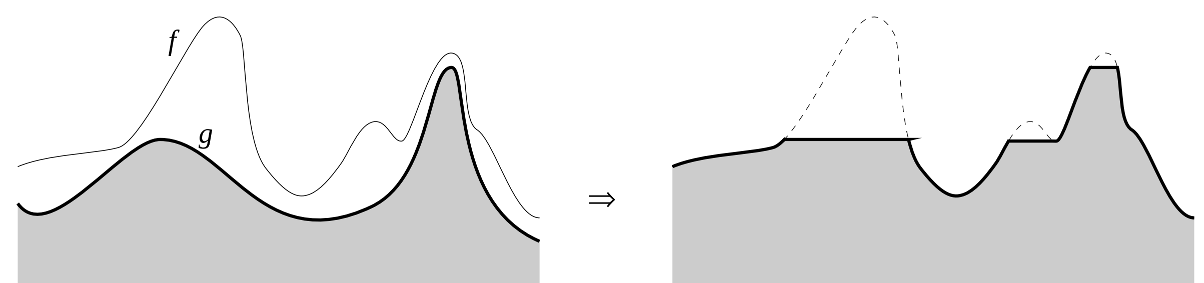


Figure 7.17: Example of decimal reconstruction.

7.3.5 Binary segmentation

We will now make use the tools that we have presented at the beginning of this chapter for designing a powerful binary segmentation algorithm. Starting from the markers of our objects, i.e. from the ultimate erosion, our goal is to contour finely these objects. We could consider using the geodesic SKIZ, and defining each object as the geodesic influence zone of its marker inside the initial set. Unfortunately, this is not a satisfactory algorithm. Indeed, as shown in Fig. 7.18, the separating lines thus defined between objects are poorly located. This is due to the fact that we did not take

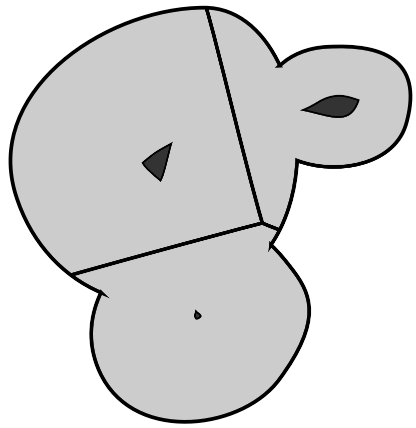


Figure 7.18: Bad segmentation algorithm (geodesic SKIZ of the ultimate erosion of X inside X).

the *altitude* of the markers—i.e. the value that is associated with them by the quench function—into account.

The way for designing a good segmentation procedure—in taking the above altitudes into account—is to use successive geodesic SKIZ. Let n_m be the size of the largest non empty erosion of X, i.e. such that

$$X \ominus n_m H \neq \emptyset \quad \text{and} \quad X \ominus (n_m + 1)H = \emptyset.$$

Necessarily, $X \ominus n_m H$ is a subset of the ultimate erosion of X (in fact, according to the notations introduced in § 7.2.1, $X \ominus n_m H$ is exactly the set $S_{n_m}(X)$). Denote X_{n_m} this set. Now, consider the erosion of size $n_m - 1$ of X, i.e. $X \ominus (n_m - 1)H$. Obviously, the following inclusion relation holds:

$$X_{n_m} \subseteq X \ominus (n_m - 1)H.$$

Now, let Y be a connected component of $X \ominus (n_m - 1)H$. There are three possible inclusion relations between Y and $Y \cap X_{n_m}$:

1. $Y \cap X_{n_m} = \emptyset$: in this case, Y is another connected component of $\mathrm{Ult}(X)$.
2. $Y \cap X_{n_m} \neq \emptyset$ and is connected: here, Y is used as a new marker.
3. $Y \cap X_{n_m} \neq \emptyset$ and is not connected: in this last case, the new markers that are produced are the geodesic influence zones of $Y \cap X_{n_m}$ inside Y.

These three different cases are shown on Fig. 7.19.

Let $X_{n_m - 1}$ be the set of the markers produced after this step. To summarize what we have just said, $X_{n_m - 1}$ is made of the union of

- the geodesic influence zones of X_{n_m} inside $X \ominus (n_m - 1)H$,
- the connected components of $\mathrm{Ult}(X)$ whose *altitude* is $n_m - 1$.

This procedure is then iterated at levels $n_m - 2$, $n_m - 3$,etc... until level 0 is reached. In a more formal way, for every $n \in]0, n_m[$, let us introduce the following notations:

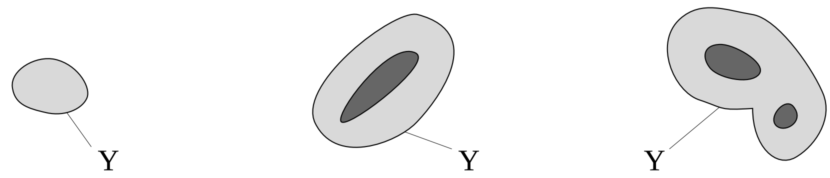


Figure 7.19: The three possible inclusion relations between Y and $Y \cap X_{n_m}$.

(i) $u_n(X)$ is the set of the connected components of $\mathrm{Ult}(X)$ having altitude n:

$$p \in u_n(X) \quad \Longleftrightarrow \quad p \in S(X) \text{ and } q_X(p) = n.$$

(ii) For every set $Y \subseteq X$, $z_X(Y)$ designates the set of the geodesic influence zones of the conected components of Y inside X.

The recurrence formula between levels n and $n-1$ can now be stated:

$$X_{n-1} = z_{X \ominus (n-1)H}(X_n) \cup u_{n-1}(X). \tag{7.22}$$

It is illustrated by Figure 7.20.

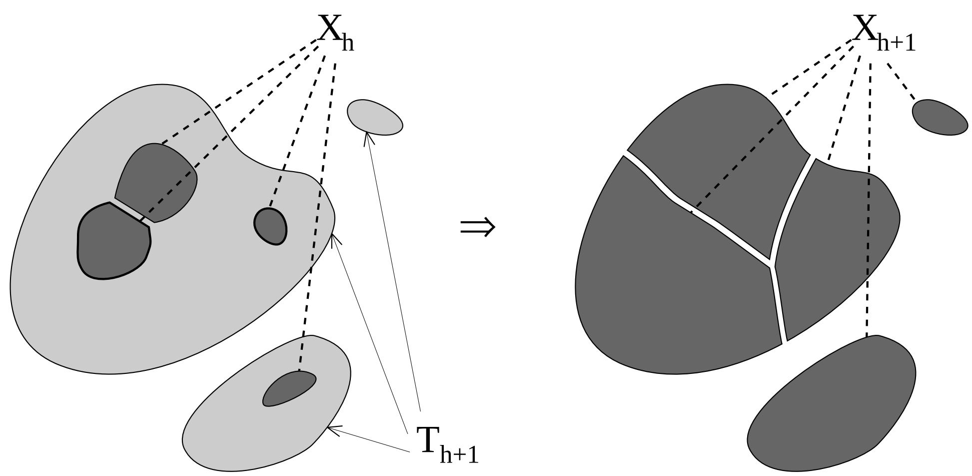


Figure 7.20: How to obtain X_{n-1} from X_n.

The set X_0 that is finally obtained after applying this algorithm produces a good segmentation of X. Fig. 7.21 presents an example of this binary segmentation algorithm. In some cases, it is still unsufficient for obtaining a satisfactory segmentation, and other procedures, making use of an a priori knowledge on the objects to segment, or based on more elaborated notions, such as *critical balls* [3], must be designed.

7.4 Watersheds and segmentation of grey-tone images

7.4.1 Catchment basins, watersheds

At first sight, the elaboration of the algorithm presented in the preceding section seems complicated. Therefore, we now give a more intuitive approach of this procedure. Consider the function $-\mathrm{dist}_X$,

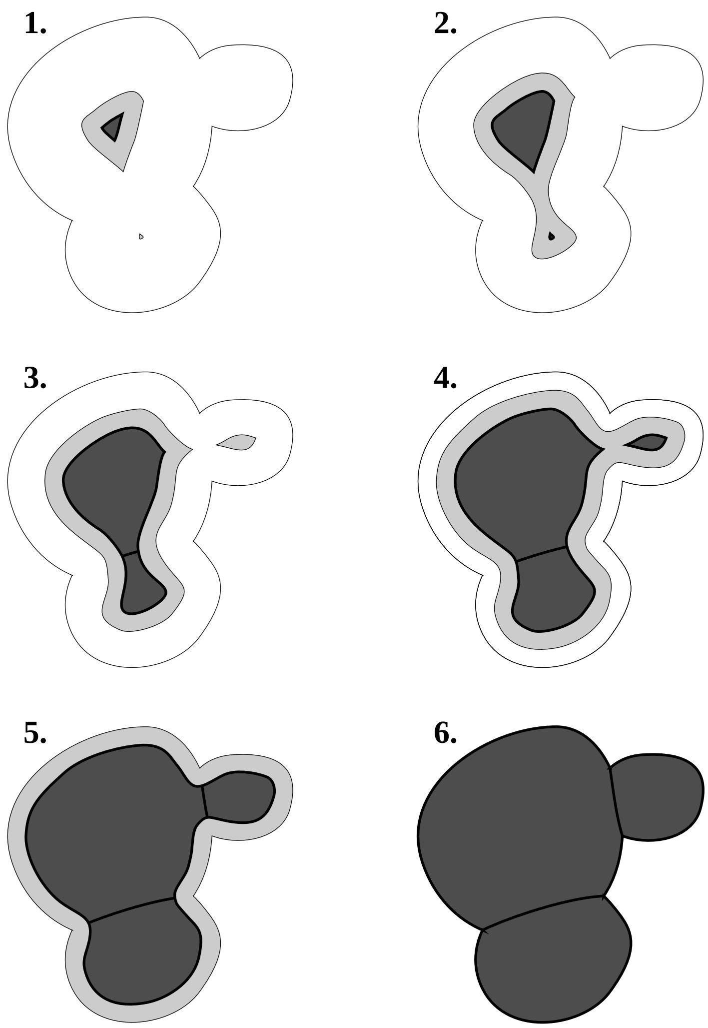


Figure 7.21: The good binary segmentation algorithm presented in § 7.3.5.

where dist_X is the distance function introduced in § 7.2.4, and regard it as a topographic surface. The *minima* of this topographic surface are located at the different connected components of the ultimate erosion of X. Now, if a drop falls at a point p of X, it will slide along the topographic surface until it finally reaches one of its minima. We define the *catchment basin* $W(m)$ associated with a minimum m of our topographic surface in the following way:

Definition 7.9 *The catchment basin $W(m)$ associated with a minimum m of a function regarded as a topographic surface is the locus of the points p such that a drop falling at p finally reaches m.*

This definition is not very formal, but it has the advantage of being relatively intuitive. In our example, the catchment basins of the function $-\text{dist}_X$ exactly correspond to the regions that were extracted by the algorithm presented in § 7.3.5.

Actually, this notion of catchment basin can be defined for any kind of grey-tone function. Moreover, the algorithm of § 7.3.5 can be easily adapted to the determination of the basins of any decimal image I: it suffices to replace the successive erosions $X \ominus nH$—which correspond to the different thresholds of the distance function of X—by the successive thresholds of I. The lines separating different basins are called *watersheds* or *dividing lines.* These notions are illustrated in Fig. 7.22.

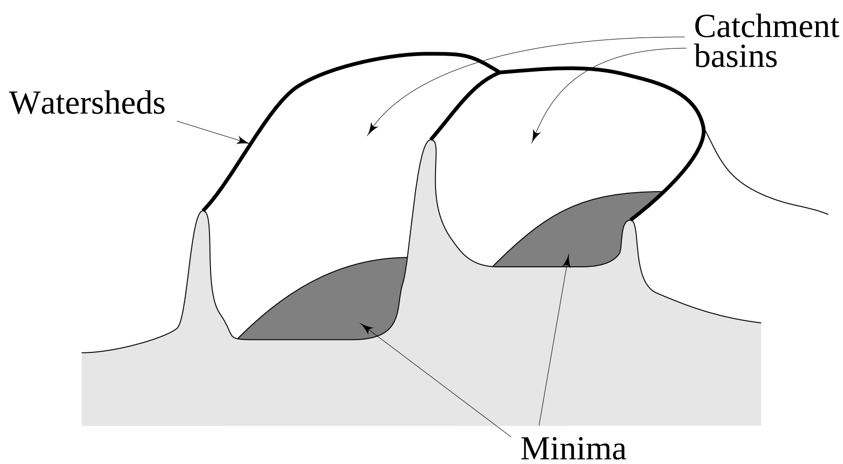


Figure 7.22: Catchment basins and watersheds

The watersheds constitute an extremely powerful tool for segmenting grey-tone images [2]. Indeed, grey-tone segmentation mostly comes down to a contour detection problem, which can be approached by watersheds: contours can be defined in grey-tone images as regions where the grey values are varying very fast, i.e. as *crest-lines of the gradient.* Notice that the gradient of a decimal image I can have several morphological definitions, the most common among them being:

$$\text{grad}(I) = (I \oplus H) - (I \ominus H).$$

The determination of the crest lines of $\text{grad}(I)$ can be done by means of the watersheds transformation. Finally, we define the contours of a grey-tone image I as the **dividing lines of its gradient.**

7.4.2 Geodesic watersheds

The watersheds of the gradient build a very general approach of contour detection. However, the resulting images are most of the time **over-segmented,** i.e. the relevant contours are swamped by a mass of irrelevant ones. This is often due to the fact that the images under study are noisy. Moreover, this approach is unsatisfactory in the sense that it does not make use of any markers when in fact we know that the first step of every segmentation is the marking of the objects to be segmented.

This initial marking step actually makes use of an external knowledge about the image or the collection of images under study. We may well want to extract only one type of objects among all those that are present. To achieve this, we shall first *mark* our objects—i.e. make use of the particular knowledge available on the problem—by means of a procedure that can be completely different from a problem to another. The question that arises then is the following: is it possible, starting from these markers, to detect the precise contours of our objects and to avoid at the same time the appearence of irrelevant contour arcs? The response is yes. By using markers, we will not remove the irrelevant contour arcs of the watersheds of the gradient, but we will avoid the over-segmentation by **modifying the gradient function** on which the watersheds are computed.

Let now I be a grey-tone image and suppose that the desired markers have been extracted. Denote $M \subset \mathbb{Z}^2$ this set of markers. They must correspond exactly to the minima of the function $\theta(I)$ on which we plan to compute watersheds. Moreover, the second requirement is that this function must be as close as possible to the gradient function $\mathrm{grad}(I)$. It is only under this condition that its dividing lines will be properly located. Therefore, starting from $\mathrm{grad}(I)$, the construction of $\theta(I)$ is done in two steps:

- Impose as minima the previously extracted markers (i.e. the set M).
- Suppress the undesirable minima.

In step 1, we simply construct the function f defined by:

$$\forall p, f(p) = \begin{cases} c & \text{when } p \in M, \\ \mathrm{grad}(I)(p) & \text{otherwise,} \end{cases}$$

with c being an arbitrary constant, strictly minorating $\mathrm{grad}(I)$.

In the step 2, we have to suppress the unwanted minima of f, without forgetting to fill their associated basins! To do so, we first construct the following function g:

$$\forall p, g(p) = \begin{cases} c & \text{when } p \in M, \\ A & \text{otherwise,} \end{cases}$$

with A being an arbitrary constant majorating $\mathrm{grad}(I)$. Then, we iteratively erode g geodesically "over" f until stability is reached. These two operations are illustrated in Fig. 7.23. The resulting function, $\theta(I)$, is such that its watersheds correspond exactly to the desired contours. Note that this transformation ($\mathrm{grad}(I) \longmapsto \theta(I)$) is actually a $\vee$-filter, since it is the composition product of a closing by an opening!...The whole procedure presented above may be referred to as *geodesic watersheds segmentation.* It is extremely powerful in a number of complex segmentation cases, since the only problem (which can be itself very complicated!) comes down to detecting the markers of the objects to extract.

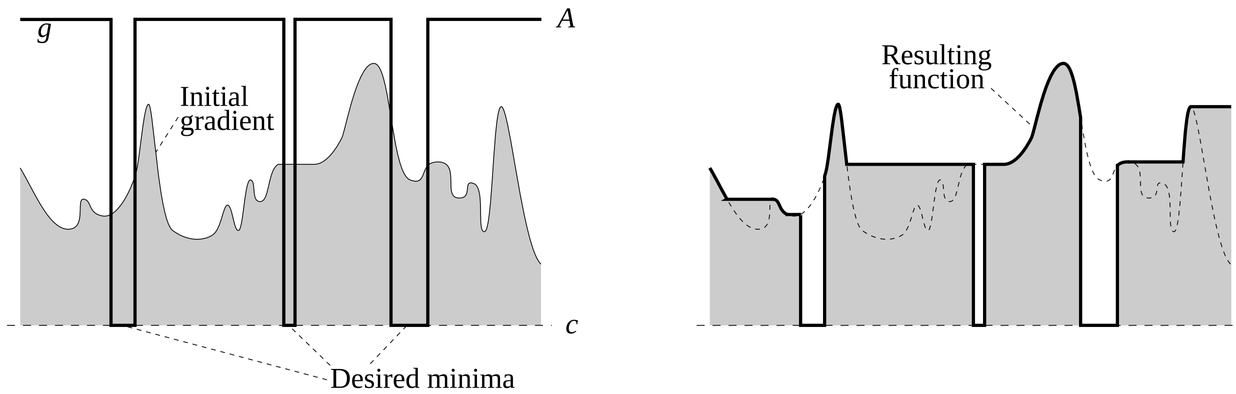


Figure 7.23: Construction of the function whose watersheds correspond to the desired contours.

7.4.3 Recent developments

The general algorithm detailed in the preceding section is based on the detection of markers of the objects present in the grey-tone image I under study. In simple cases, these markers are merely the minima of the gradient of I. Indeed, an object O often corresponds to a region of I which is relatively homogeneous compared to its neighborhood. Therefore, this region is nothing but a minimum of the gradient grad(I). The associated basin "extends" this region until it is bounded by crest lines of the gradient, i.e. by the actual contours of O.

For more complex problems, taking the minima of the gradient image of I is far from providing a good set of markers. As explained above, this may be due to noise or to the fact that the desired objects constitute only a small subset of the objects present in the image. In such cases, special marking procedures have to be designed before applying the segmentation algorithm of § 7.4.2.

But in some other cases, it is impossible to find markers of the regions or of the objects to extract, since these objects or regions are not themselves well defined. This kind of situation often occurs, for instance, with remote sensing images, where the large variety of zones (fields, roads, houses, towns, lakes,etc... under different lighting conditions) makes it almost impossible to design good marking procedures. Therefore, other segmentation algorithms have to be applied. One of the possibilities is to use the following approach:

- Computation of the watersheds of grad(I). This results in an awfully over-segmented image.
- Removal of the irrelevant contours.

This kind of method is part of a more general class of algorithms called *region growing algorithms:* Starting from the watersheds image $W(\text{grad}(I))$, we can assign to each of its different basins a value characterizing them (e.g. the mean value of the corresponding pixels in I), and produce this way a sort of *mosaic image.* In a second step, adjacent basins may be progressively merged into larger regions (thus removing contours) until a given criterium is fulfilled.

Many different criteria for merging regions can be found in literature. When associated with morphological treatments, some of them may result in particularly good segmentations [1]. Another way of approaching this problem is to regard the mosaic image of the catchment basins as a planar graph, whose vertices are the different regions and whose edges are each pair of adjacent regions (See Fig. 7.24.). This kind of object being a lattice, it can be processed by mathematical morphology [23]. Morphological merging procedures, based on gradients and watersheds on graphs [24], seem then to provide very efficient segmentation methods.

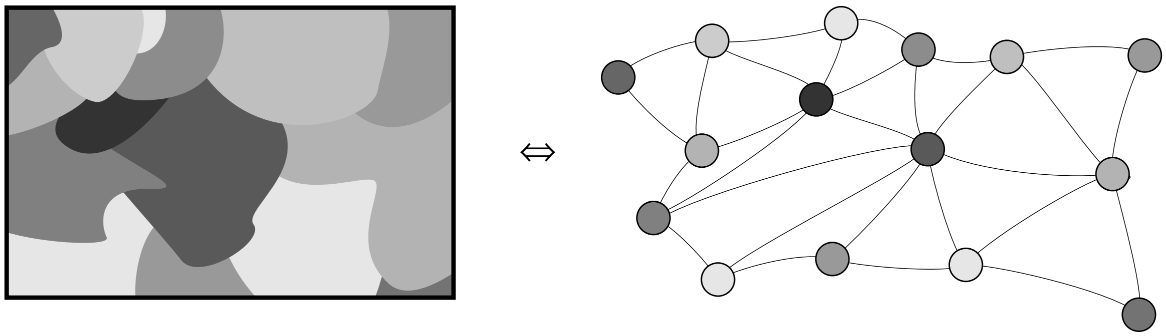


Figure 7.24: Mosaic image and associated adjacency graph.

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